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Electrical properties of coils

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ELECTRICAL PROPERTIES OF COILS

by

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INTRODUCTION

One of the chief factors in the phenomenal growth of the science of electricity and magnetism has been the relative simplicity with which the laws of this science have been formulated. For example, the laws of electric circuits can be stated with almost the same degree of exactness as the theorems of geometry. The distribution of electric currents and potentials in an electric circuit is determined by the arrangement of various materials, their electrical and magnetic properties, and a complicated set of boundary conditions. For direct currents and low-frequency alternating currents, the electrical and magnetic properties (excluding ferromagnetic materials) and the boundary conditions are relatively simple and nearly constant. Consequently, definite values of resistance, inductance, and capacitance can be assigned to particular portions of an electric circuit, each portion being known as a circuit element. Moreover, since these circuit elements have finite physical dimensions, the values of resistance, inductance, and capacitance assigned to these elements are known as lumped constants or lumped parameters. For alternating currents of higher frequencies these lumped parameters are no longer nearly constant but functions of frequency; also, the idea of lumped parameters must yield to the more tenable concept of distributed parameters. At very high frequencies the concepts of electric and magnetic fields must take complete precedence over the lumped-circuit concepts. Thus, it can be seen that the properties of electric circuits are not fundamental in themselves but are very practical equivalents of the electromagnetic field configurations associated with useful electrical components.

In order that the proper perspective be established for the object of this thesis, namely, an investigation of the electrical properties of coils at higher frequencies, a review of circuit concepts and their validity will first be given. This review will be followed by a careful definition of the coil problem and a consideration of previous techniques developed for determination of conductor losses as a function of frequency. Finally, a description of experiments performed and results obtained will be given.

CIRCUIT CONCEPTS AND THEIR VALIDITY

Simultaneous with the formulation of the circuit concepts and relations was the development of the ideas of space and time relationships between electric and magnetic phenomena. With a precision and scope that far exceeded the limits of experimental investigations of his period, James Clerk Maxwell formulated the basic laws of electric and magnetic fields.[1] Experience in nearly all phases of electrical science has confirmed these laws—their exactness and their well-nigh universal scope. Unlike the laws of electric circuits, the field laws, which are known as Maxwell's equations, have proved valid throughout the range of frequencies explored to date; in fact, it has been shown that the circuit laws are derivable from the field equations.

The ordinary circuit concepts may be summarized as follows:

$$e = Ri \,, \tag{1a}$$

$$e = L\frac{di}{dt} \,, \tag{1b}$$

$$i = C\frac{de}{dt} \,, \tag{1c}$$

where e and i are instantaneous values of potential and current, respectively, t is time, and R, L, and C are parameters known as resistance, inductance, and capacitance, respectively. In general, every element of infinite space, including all substances, is characterized by all three parameters but to greater or less extent and in different proportions. It is the fortunate circumstance that most practical circuits are composed of elements in which these parameters are separable by one means or another.

The fundamental assumptions inherent in the definitions of resistance, inductance, and capacitance are not nearly so well known as are the parameters themselves; in fact, the definitions were the result of basic experiments conducted at zero or low frequencies. Only when Maxwell's equations were applied to electric-circuit theory were the fundamental assumptions inherent in the circuit concepts fully realized. The earliest published paper on this subject was written by J. R. Carson.[2] Since then, at least two texts on electromagnetic phenomena have been so written as to include a treatment of these assumptions.[3, 4] The first basic assumption, namely, that the velocity of electromagnetic propagation is infinite, has several consequences that are sometimes regarded as further assumptions. Some of these consequences are complete absence of electromagnetic radiation, complete independence of the current in a conductor of the distance along the conductor parallel to the current flow, and an infinite wavelength of the electromagnetic disturbance. The second fundamental assumption may be stated two ways, either that all currents are filamentary in character or that there are no proximity or skin effects on currents in conductors of finite cross section. Finally, the third fundamental postulate is that there is no coupling by means of electromagnetic fields between parts of a circuit or circuits except in well-defined local regions, such as mutual inductances between coils.

The process of modifying Maxwell's field equations in accordance with the aforesaid assumptions is too involved to be described briefly yet intelligibly in a few pages. It is sufficient to say that Carson,[2] King,[3] and Ramo and Whinnery[4] have treated this problem very admirably. What is really pertinent to this thesis, however, is the practical significance of these postulates with respect to circuit theory, both in general and with particular emphasis on coils used at audio and higher frequencies.

For direct-current networks all these basic assumptions are exact, and the network parameters (resistance, inductance, and capacitance) are wholly unique. In other words, each parameter is a function only of the material and geometry of the circuit component to which that parameter applies and is, except for mutual impedances, independent of the way in which that circuit component is connected in a circuit and of the orientation of the circuit component with respect to its environment, which is assumed to consist of stationary fields only. The practical significance of this uniqueness of the circuit parameters is that resistances, inductances, capacitances, and even mutual impedances for direct-current networks can be designed and given specific numerical labels without regard to the network in which these parameters are employed. In other words, a network can be conceived as being composed of lumped constants. For alternating-current networks these fundamental assumptions are no longer exact but are approximate. At very low frequencies these approximations are very good, but as the frequency scale is ascended these approximations become poorer and poorer until they become wholly untenable. However, the usefulness of the simplified theory based on networks with lumped constants makes desirable the modification of the circuit parameters, recognizing the latter as functions of frequency. The phenomenon of skin effect is handled in this manner; the concepts of distributed capacitance of a coil and of distributed constants of a transmission line are typical examples of modified parameters. In general, adjustment of the circuit parameters as the frequency is increased is accomplished by reducing the number of approximations and by reducing the scope of those that are still retained.

It is considered appropriate at this point to examine the validity of these fundamental postulates when applied to alternating-current systems. The first basic assumption, which concerns the velocity of propagation, affects the phase shift of potentials and currents in the various parts of a circuit and consequently the propagation of energy. Thus, the potentials and currents at the receiving end of a long transmission line will not, in general, be in phase with the corresponding quantities at the sending end but will differ in phase by an amount proportional to the time of propagation from the one end to the other. If the velocity of propagation is v and the generator frequency f, the time lag at a distance x from the generator is $\frac{x}{v}$ (in time units), and the phase shift is given as

$$\alpha = \frac{fx}{v} \text{ cycles}, \qquad (2a)$$

$$\alpha' = \frac{2\pi fx}{v} \text{ radians}, \qquad (2b)$$

$$\alpha'' = \frac{360\ fx}{v}\ \text{degrees}\ . \qquad (2c)$$

The velocity v is usually close to that of light, It is seen from these expressions that the upper limit of frequency depends on both the distances along a circuit and the permissible error caused by this propagation time. For example, if the maximum allowable phase error is 1 degree and the longest circuit dimension is 1 meter, then, the frequency range over which the first postulate is valid throughout the circuit is from 0 to 830 kilocycles per second. The shorter parts of the circuit have a correspondingly higher permissible frequency range. This fact means that the first basic assumption will remain valid over parts of a circuit after the upper frequency limit is exceeded for other circuit parts. Consequently, it is permissible to have a hybrid mixture of low-frequency approximations and high-frequency refinements in the treatment of a single circuit. A more important deduction from Equation 2 is the fact that the frequency range becomes limitless when circuit dimensions become infinitesimal. The practical significance of this consequence is the evolution of the concept of distributed parameters from the concept of lumped parameters.

A second illustration of the investigation of the range of frequencies over which the first fundamental assumption may be considered valid is the investigation of radiation from an antenna.[4] Table 1 and Figure 1 show the comparative variations of conductor and radiation resistances for a differential antenna 1 meter long and made of Number 2 (American Wire Gauge) round copper wire. As can be verified by Figure 1, the radiation

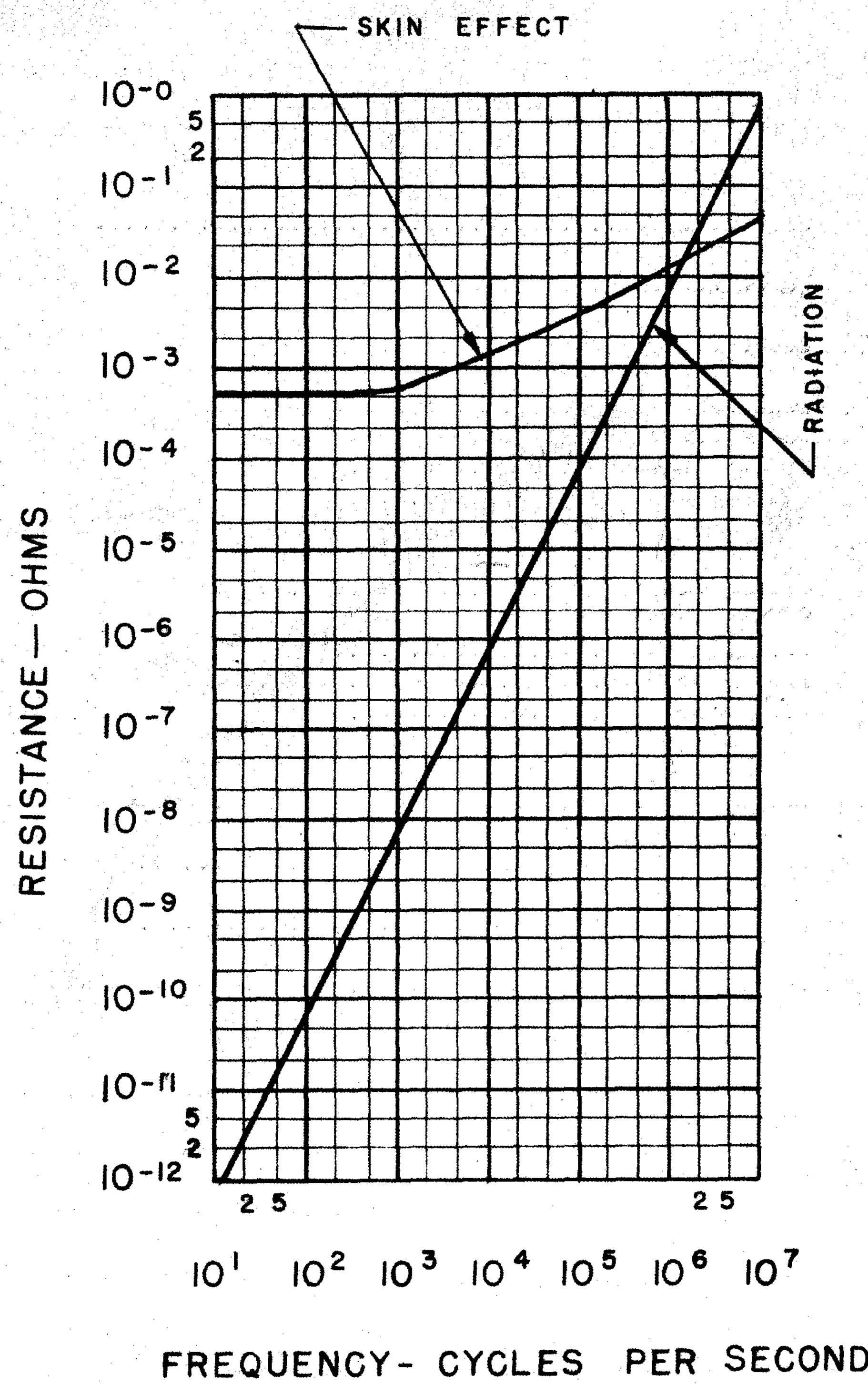
Table 1. Comparison of Radiation and Skin Effects

Frequency (cycles per second)	Radiation Resistance (ohms)	Conductor Resistance (ohms)
10^7	0.877	0.04096
10^6	8.77×10^{-3}	0.01304
10^5	8.77×10^{-5}	0.004217
10^4	8.77×10^{-7}	0.001431
1000	8.77×10^{-9}	0.000581
100	8.77×10^{-11}	0.000522
10	8.77×10^{-13}	0.000522
0	0	0.000522

resistance is 1 per cent of the conductor resistance near 63 kilocycles per second, is 10 per cent near 298 kilocycles per second, and is equal to the conductor resistance near 1.3 megacycles per second.

The validity of the second basic assumption, which concerns skin and proximity effects, is a function of not only the frequency but also the electrical resistivity, magnetic permeability, and physical dimensions. For example, copper conductors of circular cross sections but of unequal diameters will not exhibit the phenomenon of skin effect to the same degree at the same frequency; in fact, the larger wire will exhibit more skin

Figure 1. Comparison of Radiation and Skin Effects.


SKIN EFFECT
RADIATION
RESISTANCE — OHMS
10-0
5
2
10-1
10-2
10-3
10-4
10-5
10-6
10-7
10-8
10-9
10-10
10-11
5
2
10-12
2 5
2 5
10¹ 10² 10³ 10⁴ 10⁵ 10⁶ 10⁷
FREQUENCY- CYCLES PER SECOND

effect than the smaller. Likewise, an iron wire will have more skin effect than a copper wire of the same size and used at the same frequency. If x be defined[4] as

$$x = r\sqrt{\omega \mu \sigma}\,, \qquad (3)$$

where r is the wire radius, ω the angular frequency, μ the absolute magnetic permeability of the wire, and σ the electrical conductivity of the wire, it can be shown that a variation in x from 0 to 1.2 will result in an increase of 1 per cent in the resistance of an infinitely long conductor of circular cross section.* Because x is an important dimensionless parameter in the analysis of skin and proximity effects, it is proposed that this quantity be called the "skin-effect number". The magnitude of the proximity effect is a function of all these things and of the conductor separation.

Finally, there remains to be considered the validity of the third basic assumption, according to which electromagnetic coupling is negligible everywhere except in well-defined regions. The first consideration is a series of facts pertinent to the subject of coupling. (1) There is an electric field associated with every charge and with every changing magnetic field. (2) There is a magnetic field associated with every current and with every changing electric field. (3) These electric and

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*The rationalized MKS system of units is used throughout this thesis. If the electromagnetic system of units is used, the right side of Equation 3 must be multiplied by $2\sqrt{\pi}$.

magnetic fields extend to infinity theoretically, though practically they become negligible at finite distances from their respective sources. (4) Both electric and magnetic fields consist of two significant components; one is the induction component and varies inversely as the square (magnetic) or cube (electric) of the distance from its dipole source, and the other is called the radiation component and varies inversely as the first power of the distance from its dipole source.* At a distance of $\frac{1}{2\pi}$ of a wavelength from their dipole source both components are equal, the induction component being the larger of the two at shorter distances and the radiation component likewise at longer distances. Thus, at very low frequencies (or very long wavelengths) the induction component predominates to the almost complete exclusion of the radiation component in most practical circuits. (5) The strength of the magnetic field very close to the dipole source (much less than $\frac{1}{2\pi}$ of a wavelength distant) is nearly independent of the frequency of the source, while the strength of the electric field in this region varies approximately inversely as the frequency of the source. On the other hand, at points remote from the dipole source the strengths of both electric and magnetic fields vary almost directly as the frequency of the source.

The facts just listed as well as others show that while electromagnetic coupling exists between every pair of points in space, whether free or occupied, for practical purposes such coupling is negligible

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*See page 430 in Reference 4.

except between points not too far apart. This statement must be vague because of the arbitrary conditions involved. The rate of change of concentration of charge or of current or of both at a point determine the strength of signal emitted therefrom; how far away from this first point a second point can be and still detect the signal depends not only on the attenuation of the signal over the distance traveled but also on the minimum strength of signal that can be detected at the second point. Hence, there is no sharp line of demarcation beyond which it can be said no signal can be detected, and the same is true for frequency. Another important practical consideration is the relative strengths of signals being received at a point from all other points. Thus, a relatively few signals received at a point can have such strength as to cause all other signals received to be negligible by comparison.

In conclusion, it can be said that a rigorous application of Maxwell's field equations shows circuit concepts to be untenable except for their great aid to practical electromagnetic problems and then only where the approximations upon which these concepts are based are reasonable. Without the aid of these and other approximations quantitative use of Maxwell's equations would be prohibitive. Therefore, in any electromagnetic problem it is important to know the number and scope of approximations that can be made in the light of results to be attained. For the purposes of this thesis the validity of these approximations for frequencies between 10 and 100,000 cycles per second (corresponding to wavelengths between 3 and 30,000 kilometers) will be of chief concern.

The dimensions of most low-frequency circuits are small fractions of the shortest wavelength to be considered; especially is this comparison true for the coils whose properties are to be investigated herein. Hence the first basic assumption for electric circuits, namely, that the velocity of electromagnetic propagation is infinite, is accepted as valid for the scope of this thesis. The second and third fundamental postulates for electric circuits are not accepted as valid, since skin and proximity effects and mutual coupling of neighboring circuits are the chief factors in the variation of the electrical properties of coils throughout the frequency range chosen for this investigation. In fact, the major portion of this thesis will be the investigation of skin and proximity effects and of the effect of mutual coupling on the electrical properties of coils.

THEORETICAL APPROACH

Basic Relations

As has been stated in the preceding sections, the chosen scope of this thesis is the study of the relation between the electrical properties of coils and frequency in the audio range. In this particular region of the frequency spectrum the wavelengths are very long compared to the dimensions of most practical coils; hence, it is justifiable in this instance to regard the velocity of electromagnetic propagation as infinite. Since this velocity v is related to magnetic permeability μ and dielectric constant or permittivity ϵ by the relation

$$v = \frac{1}{\sqrt{\mu\epsilon}}, \tag{4}$$

the implication is that either μ or ϵ is zero. Without a finite μ there would be no magnetic field. Accordingly, Maxwell's equations as modified for the convenient analysis of skin and proximity effects are (in vector notation)*

$$\nabla \times \bar{H} = \bar{i} = \sigma \bar{E}, \tag{5a}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t}. \tag{5b}$$

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*A horizontal bar above a letter denotes a vector quantity, where the same letter without the bar denotes a scalar quantity.

The quantity $\bar{H}$ is defined as the magnetic intensity, $\bar{E}$ as the electric intensity, $\bar{B}$ as magnetic flux density, $\bar{\iota}$ as electric current density, and σ as electrical conductivity. All quantities except the last are vector quantities. The first equation above also includes Ohm's law. Both equations govern all skin-effect phenomena in conductors to the extent that charging currents in dielectrics can be neglected.

As has been shown by Ramo and Whinnery,[4] Equations 5a and 5b can be manipulated so as to yield the following relations:

$$\nabla^2 \bar{H} = \sigma \mu \frac{\partial \bar{H}}{\partial t}, \tag{6a}$$

$$\nabla^2 \bar{E} = \sigma \mu \frac{\partial \bar{E}}{\partial t}, \tag{6b}$$

$$\nabla^2 \bar{\iota} = \sigma \mu \frac{\partial \bar{\iota}}{\partial t}. \tag{6c}$$

These equations are sometimes known as the skin-effect or distribution equations. For sinusoidal time variations of $\bar{H}$, $\bar{E}$, and $\bar{\iota}$ these equations become

$$\nabla^2 \bar{H} = j \omega \sigma \mu \bar{H}, \tag{7a}$$

$$\nabla^2 \bar{E} = j \omega \sigma \mu \bar{E}, \tag{7b}$$

$$\nabla^2 \bar{\iota} = j \omega \sigma \mu \bar{\iota}. \tag{7c}$$

Methods of Approximation

Different methods of approach for solution of skin-effect problems have been found in the literature. Nearly all of them begin with the basic relations just defined. Because the chief problem in any investigation of these phenomena is the application of boundary conditions, the different methods can be sorted largely on the basis of number and type of simplifying assumptions. The following review is an attempt to sort the methods applied to coils.

Sommerfeld[5] in 1904 and Coffin[6] in 1906 approximated an m-layer coil by m concentric cylindrical current sheets. The coil was assumed to be infinitely long, so that the magnetic field strength and current distribution were functions of only the distance from the coil axis. Later investigators showed these theoretical developments to be inadequate.

In 1907 Cohen[7] attacked the problem of the single-layer coil. Except for the restriction to one layer, Cohen's analysis differed from the earlier ones by the assumption of a coil wound with square wire with zero spacing. Cohen, however, did ignore the helical configuration of the winding. He checked his formulas against a few experimental data of his own and of others and found fairly good agreement. His optimism was unjustified, however, since the tests were very limited both in number and in range. Later investigations by others confirmed the inadequacy of Cohen's formulas.

Over a decade passed before further studies of the problem of a-c coil resistance were published. Only those which were fundamental will be cited. In 1920 G. W. O. Howe[8, 9] applied transmission-line techniques to coils. His theoretical approach, which appears to be original, is equivalent to the more widely recognized basic approach. As in all other cases described in this thesis, Howe's work is to be distinguished by his choice of boundary conditions. At lower frequencies he assumed a single-layer coil wound with rectangular wire to be a transmission line for radial flow of energy. Omission of energy flow other than radial is equivalent to the often-used concept of current sheets. Howe improved upon this concept, however, by assuming the relative permeability to be the ratio of the axial dimension of the square wire to the coil pitch instead of unity; thus, he compensated partially for loosely wound coils. Coils with round wire were approximated by coils of square wire of equal cross section.

Howe's treatment of m-layer coils was rather novel. For infinitely long coils he assumed the magnetic field strength to vary stepwise in the proportion of 0, 1, 2, ---, m as one counts the air spaces beginning with the one outside the coil and progressing inwardly from one annular space to the next until the innermost air space is reached. Consistent with this assumption, he divided the annular cross section of each turn into two annular halves and assumed currents to be flowing in each annular portion in the same proportion as the strengths of the respective adjacent fields. The currents in the two halves of any turn were assumed to flow in opposite directions, the current in the inner half being in the same direction as the prevailing coil current. Thus, a survey of the coil current distribution, starting with the outer layer and moving toward the axis, would show the following:

Layer	Current
1	0 I
2	-I 2I
3	-2I 3I
⋮	⋮
m	-(m-1)I mI

The foregoing assumptions were the boundary conditions for the treatment of the coil as a series of radial transmission lines, one for each layer, each line except the outermost being supplied with power from both ends and the outermost line from one end only. This technique is very elegant but wholly neglects the radial components of the magnetic field of the coil.

For short single-layer coils Howe used the results for long coils except that the field strength on the inner side of the central turn was corrected to what it would be for the short coil. This correction factor was applied to the already modified relative permeability. Both modifications of the relative permeability are legitimate except insofar as they fail to include the complicated effects of the eddy currents on the field distribution.

Howe followed his theoretical work with experiments, using the Poulsen arc as an exciter and a calorimeter scheme. In the light of methods and apparatus now available, his experimental results fall short of being reliable. Hence, the discrepancies between his calculations and data were really too great to be simply explained. It may be said in conclusion that Howe's chief contribution to the problem of coil analysis was the transmission-line technique.

What is believed to be the outstanding contribution to the solution of the problem of the resistance of coils excited by alternating current is the work of Butterworth[10, 11, 12] as stimulated by that of Hickman[13] and supplemented by that of Arnold.[14] In 1922 Butterworth[10] gave an exact analysis (except for neglect of displacement currents) of the eddy-current losses in an infinitely long non-magnetic conducting circular cylinder exposed to an alternating magnetic field perpendicular to and invariant with respect to points along the axis of the cylinder. Using Poynting's theorem, Butterworth determined the rate of energy flow into the cylinder for a unit length. Because of its fundamental importance this development is given in Appendix A. Since the magnetic field was general except for the restrictions mentioned above, Butterworth's results up to this point were equally general. He applied his results to the case of a cylinder exposed to the field of a current-carrying thin wire laid parallel to the axis of the cylinder; then, this case was extended to that of two thin wires equally and oppositely spaced with respect to the cylinder so that the three axes were parallel and coplanar. The two wires were assumed to carry current both oppositely and in the same direction.

Butterworth next considered the analysis of eddy-current losses in two parallel equal cylinders carrying equal currents both oppositely and in the same direction. He showed that the eddy-current losses could be resolved into three components. The first component was found to be identical to the skin-effect losses of an isolated conductor and is expressed as

$$\tfrac{1}{2}R_{d-c}I^2 \left[1 + F(x)\right] ,$$

where R_{d-c} is the resistance of the conductor to direct current, I is the peak value of the alternating current, and x is defined in Equation 3. Values of F(x) and 1 + F(x) have been tabulated in the literature.[12,15,16] The second component was found to be what is called the first-order proximity effect when the two conductors are relatively far apart and is given as

$$\tfrac{1}{2}R_{d-c}\, I^2 \left[\left(\frac{2a}{D}\right)^2 G(x)\right] ,$$

where a is the conductor radius and D the axial separation of the two cylinders. Values of the function G(x) have been tabulated in the literature.[12, 16] When x is greater than 5, the functions F(x) and

$G(x)$ may be fairly represented by their asymptotic expressions as follows:

$$F(x) \doteq (\sqrt{2}\,x - 3)/4\,, \qquad (8)$$

$$G(x) \doteq (\sqrt{2}\,x - 1)/8. \qquad (9)$$

The second component is what would have been found had the second cylinder been replaced by a uniform field. Finally, the third component was found to include "the effect both of disturbance of current distribution due to proximity and of non-uniformity of the field." Although Butterworth gave approximate expressions for this component, the conditions under which the approximations are valid are too involved to permit a brief treatment; hence, these expressions are not shown in this dissertation. It is sufficient to say that the theoretical results just described were found to be in agreement with the experimental findings of other investigators. It should be noted, however, that the third component was found to be less when the currents were similarly directed than when the currents were oppositely directed.

Butterworth applied the foregoing results to short coils with the following added assumptions:

(1) The winding section is small in comparison with the radius.

(2) The magnetic field for each individual turn is that due to the current in the remaining turns, each turn in the same layer being replaced by a thin wire and the other layers by current sheets.

(3) The radial depth of the coil is small compared to the axial length, which is small compared to the radius of the coil.

(4) The effect of helicity is negligible.

(5) The effect of curvature is important only as it concerns the magnetic field.

The results, although very interesting, are not included in this thesis, because they have been superseded by later improvements yet to be discussed. Butterworth did show, however, that in the case of single-layer coils the increase in the equivalent resistance as based on Joule's law with frequency for each turn is greater as the distance from the nearer end decreases. This fact is important when coil temperatures are measured for the purpose of determining the coil resistance. Butterworth also showed that the equivalent resistance as based on the power quantity "E dot I" for each turn varies oppositely to that based on Joule's law. In fact, the "E dot I" resistance equivalent not only is a maximum for the central turns of the coil but actually becomes negative for the end turns. This difference between the two points of view is explained entirely by the phenomenon of "transformer action" among the turns. (Quotations are the author's.)

Hickman,[13] being dissatisfied with Butterworth's approximations, attempted a novel approximation with results surprising to the author of this thesis. Hickman's approximation is described in the following manner. Imagine two parallel planes in which are arranged a number of infinitely long parallel conductors of equal diameters and equally spaced in each plane. Imagine a corresponding coil of the same wire size and pitch and having a diameter approximately twice the separation of the two parallel planes. Hickman assumed that since the two systems have nearly the same total magnetic fields and since for a length of the parallel conductors equal to half the coil circumference the inductance is nearly the same, the variation of resistance and inductance with frequency should be nearly the same for both systems. Consequently, he analyzed the system of parallel conductors, assuming equal current distribution in each, and applied the results to coils. It is apparent that for coils whose wire radius is a large fraction of the coil radius the results would not be applicable, since the analysis was based on straight wires; this fact does not exclude many practical coils, however.

Hickman gathered data on four coils made by himself and confirmed his own findings, using a series R-L bridge. He also measured the resistance of each turn and confirmed both of the theoretical distributions predicted by Butterworth. As further confirmation Hickman measured the resistance of each turn of the excited portion of a 160-turn coil when only 100 turns were excited. He found the resistance per turn to be greater than when the whole coil was excited, rising more steeply between Turn No. 10 and Turn No. 95 (numbering only the excited turns) but dropping off sharply for the remaining five turns.

Although Hickman's experimental results were in good agreement with his own formulas, he was unable to find agreement with those of Butterworth. Consequently, Butterworth published another dissertation[11] in which he showed that his formulas did not apply to Hickman's experimental results, since the latter's coils were closely wound. Butterworth extended the scope of his formulas by suitable analysis so as to include closely wound coils, but he failed to check Hickman's experimental results as closely as did Hickman himself. Butterworth's analysis is interesting, however, because of his approach to the problem. Proceeding from the basic relations established in his earlier paper, he determined "the losses in a single-layer plane system of equidistant parallel wires infinite in number" for three cases, namely, (1) equal currents, (2) exposure to a uniform field parallel to the plane of the wire axes but normal to the directions of the axes, and (3) exposure to a field normal to the plane of the axes. Then, for an infinite parallel wire system carrying equal currents in all wires and exposed to a field inclined to the plane of the system but normal to the wire axes, Butterworth showed the conductor losses to be the sum of the losses determined for each of the three cases previously mentioned. He then derived Hickman's formulas from his own by introducing approximations corresponding to Hickman's unique assumptions. This accomplishment more than offsets the experimental evidence that Hickman's formulas are better and confirms the danger of accepting agreement between theory and experiment at face value.

Despite the tremendous amount of work just described, the formulas developed by Butterworth were limited to two very restricted groups of single-layer coils, those of short length and having a finite number of turns and those of any length and having an infinite number of turns. It remained for Arnold[14] to extend these formulas to single-layer coils of intermediate design. By both theoretical considerations and empirical methods Arnold made these extensions, but he did not establish them by any rigorous procedure. His paper is largely a summary of formulas and tables and a discussion of their limitations in accuracy and scope. Confirmation of the formulas and tables was based on the comparison between calculated values and the experimental data of Medhurst[17] for 39 coils. Arnold's results were on the average $2\frac{1}{2}$ per cent below those of Medhurst. Arnold's results checked Hickman's experimental data by a root-mean-square difference of $5\frac{1}{2}$ per cent with an estimated experimental error of 5 per cent.

Arnold's formula for a coil of any length and having any number of turns is as follows:

$$\frac{R_{a-c}}{R_{d-c}} = \left[\alpha + \eta^2 \chi (\beta \mu_1 + \gamma \mu_2)\right](1 + F). \qquad (10)$$

The factor 1 + F is the skin-effect ratio for an isolated cylindrical conductor of infinite length. The term α is the proximity-effect term and is a function of x, which is defined by Equation 3, and of the ratio η of the wire diameter to the separation of turns. The quantity

χ is strictly a function of x and is related to the induction of eddy currents in a conductor by an external alternating field. The factor $\beta u_1 + \gamma u_2$ is another proximity-effect relation, the first term relating to the transverse components of the magnetic field and the second to the tangential components. All these quantities are tabulated in Arnold's paper for a range of values of coil parameters. Although the above work appears to be quite satisfactory, it is only for single-layer coils. It was considered desirable, therefore, to give more attention to multilayer coils in the remainder of this thesis.

Proposed Method of Attack

In all the approaches just described approximations were made in both the fundamentals and the specific applications of these fundamentals. Only in the work of Butterworth were the basic relationships found to be derived with acceptable approximations, the latter being wholly limited to the clearly defined assumptions. In order to avoid the enormous labor of exact calculations Butterworth applied his theory only to coils of extreme design, neglecting the coils of intermediate proportions. As indicated in the preceding section Arnold[14] extended Butterworth's applied theory by semi-empirical means to single-layer solenoids of nearly all proportions. Close agreement between Arnold's formulation and Medhurst's experiments on 39 coils[17] seems to establish the validity of Arnold's work, especially since each man accomplished his objective with utmost integrity.

It is proposed to do for multilayer coils in a limited way what Arnold did so completely for single-layer coils. For convenience a "master" coil of 4 layers and 70 turns per layer was chosen with the idea that both theoretically and experimentally many smaller coils could be obtained from the master coil. In order to avoid distributed-capacity effects the turns were loosely wound, and each turn of each layer except the innermost was wound on top of the turn directly under it rather than in the grooves between turns of the layer beneath the one being wound. The earlier choice of audio frequencies also assisted in this objective. For example, the master coil has a distributed capacitance of 257.5 micromicrofarads and an inductance of 590 microhenrys. The apparent natural frequency is over 400 kilocycles per second; hence, the highest test frequency of 20 kilocycles per second that was used in these studies is less than 5 percent of the apparent natural frequency. It follows that the equivalent resistance of the master coil and of the smaller coils obtained therefrom will be affected by less than 0.5 per cent because of distributed capacity.

For coils which are not too closely wound it is practical to assume that the magnetic field is uniform throughout any particular cross section of the wire in the winding. By the "magnetic field" is meant the field caused by the current in the remaining conductors (not including the field caused by the current in the turn under consideration). Butterworth and others have utilized this assumption in their several investigations with varying degrees of success. The chief barrier in this approach, however, has been the accurate determination of the mean square field of the coil, not because of the mathematical hurdles but due to the extremely large number of computations involved. One of the improvements Arnold made in the application of Butterworth's basic theory was a more precise determination of the mean square magnetic field for many configurations of single-layer coils. Ignoring only the helicity of the coil, Arnold computed by means of exact formulas the radial and axial components of the magnetic field on the wire axis (not the coil axis) of each turn as the result of uniformly distributed current in the remaining turns. It is proposed that this same refinement be extended to multilayer coils.

As shown in Appendix A, the average power dissipation per unit length of an infinitely long cylinder exposed to a uniform alternating magnetic field transverse to the axis of the cylinder is given by the following expression:

$$\frac{P_{av}}{l} = \frac{8\pi H^2}{\sigma} G(x). \tag{11}$$

As indicated by Butterworth, this relation is applicable to a coil which is not closely wound and whose wire diameter is small compared to the coil diameter. Both of these restrictions validate the assumption that the magnetic field is uniform throughout any particular cross section of the wire in the winding. Percentagewise the proximity effect of producing nonuniformity of the field is much less in a coil than between two parallel wires because of the strong field associated with the coil. Consequently, the validity of the above assumption extends to a closer spacing for coils than for two parallel wires. Likewise, the same may be said for coils of many turns as compared to coils of few turns. The author has been unable to determine quantitatively, however, the limits of validity of the assumption of a uniform field for a coil, but it seems by the above reasoning that this assumption is more valid for multilayer coils than for single-layer coils.

In order to apply Equation 11 to a coil, it is convenient to neglect helicity of the winding. Therefore, l may be taken to be the length of a turn and H the mean field intensity to which a turn is exposed because of the current flowing in the remaining turns. Although the curvature of the coil is recognized in the determination of the field intensity, the effect of curvature on the d-c resistance of the coil is neglected. Neglect of helicity and of non-uniformity of field and current in the wire cross section permit the field intensity to be determined as though all turns were filamentary coaxial rings.

The magnetic field intensity H at any point P due to a circular filament of current I is expressed in terms of radial and axial components as follows (see Figure 2):

$$H_r = \frac{I}{8\pi b}\sqrt{\frac{4\,ab-k^2(a+b)^2}{ab}}\left[\frac{2-k^2}{1-k^2}E - 2K\right]\frac{z}{|z|}, \qquad (12)$$

$$H_z = \frac{I\,k}{4\pi\sqrt{ab}}\left[K - \frac{2b-(a+b)k^2}{2b(1-k^2)}\;E\right]. \qquad (13)$$

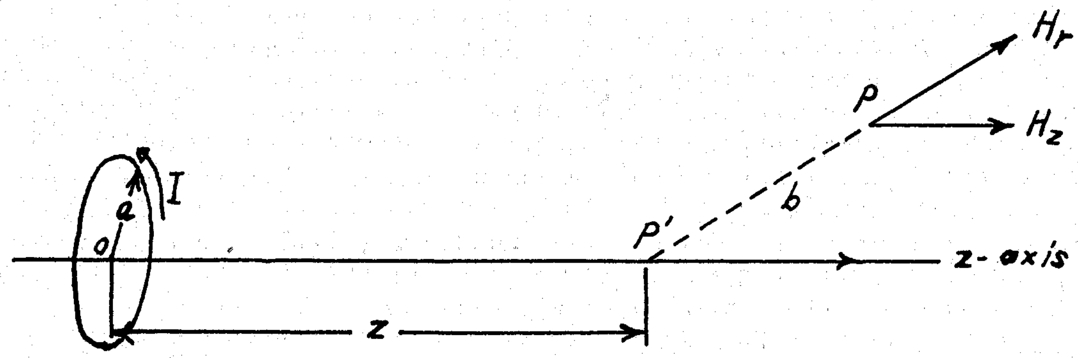


Figure 2. Coordinates for Determination of Field Intensity at a Point Some Distance from a Current Ring.

where H_r = radial component of H,

H_z = axial component of H,

K = complete elliptic integral of the first kind to modulus k,

E = complete elliptic integral of the second kind to modulus k,

$$k^2 = \frac{4\,ab}{(a + b)^2 + z^2}\ ,$$

a, b, and z are as shown in Figure 2.

The expression for power in Equation 11 may now be summed over the entire coil as follows:

$$P = \frac{8\pi}{\sigma} \; G(x) \sum_{j=1}^{m} \left(2\pi b_j \sum_{i=1}^{n} H_{ij}^2\right) \; , \qquad (14)$$

where i = turn number (counting from one end of coil),

n = number of turns per layer,

j = layer number (counting from innermost layer),

m = number of layers,

b = radius of any turn,

H_{ij} = resultant field on the wire axis of the i th turn in the j th layer due to current in all the remaining turns.

In addition to the above power loss is the skin-effect loss, which is identical to what would exist for the coil winding if it were straightened. In Butterworth's notation, the skin-effect loss is expressed as

$$P' = \tfrac{1}{2}I^2 \, R_{d\text{-}c} \left[1 + F(x)\right] \; , \qquad (15)$$

where I is the alternating-current amplitude. The direct-current resistance of the coil is

$$R_{d\text{-}c} = \frac{n}{\pi r^2 \sigma} \sum_{j=1}^{m} (2\pi b_j) \; , \qquad (16)$$

where r is the wire radius. Combining the losses P and P' and dividing by $\frac{1}{2}I^2$ yields the following expression for the total equivalent resistance

R_{a-c} of the coil:

$$R_{a-c} = R_{d-c}\left[1 + F(x)\right] + \frac{32\pi^2}{\sigma I^2} G(x) \sum_{j=1}^{m} (b_j \sum_{i=1}^{n} H_{ij}^2). \qquad (17)$$

Therefore, the resistance ratio for a coil may be expressed as

$$\frac{R_{a-c}}{R_{d-c}} = 1 + F(x) + \frac{16\pi^2 r^2}{I^2 n} G(x) \frac{\sum_{j=1}^{m}(b_j \sum_{i=1}^{n} H_{ij}^2)}{\sum_{j=1}^{m} b_j}. \qquad (18)$$

For convenience in computation this equation is rearranged to read as follows:

$$\frac{R_{a-c}}{R_{d-c}} = 1 + F(x) + r^2\, G(x) \frac{\sum_{j=1}^{m}\left[b_j \sum_{i=1}^{n}\left(\frac{4\pi}{I} H_{ij}\right)^2\right]}{n \sum_{j=1}^{m} b_j}. \qquad (19)$$

The quantity

$$\left(\frac{4\pi}{I} H_{ij}\right)^2$$

is obtained from Equations 12 and 13 by summing all the components squared as obtained for all values of z and a except for a = b when z = o; in other words, the contribution of the current in a particular turn to the field at that turn is omitted.

Finally, there are three types of symmetry that make possible a considerable reduction in the labor of computation. These symmetries are as follows:

(1) The modulus k is symmetrical with respect to a and b. Therefore, H_r (see Equation 12) is symmetrical with respect to a and b except for the factor $\frac{1}{b}$.

(2) Any two turns in the same layer that are equally and oppositely spaced from the turn whose field is being determined, whether in the same layer as this turn or not, contribute equal values of H_r but of opposite sign and equal values of H_z of the same sign.

(3) Half of the coil (as taken from one end to the middle) is the image of the other half.

Experimental difficulties permitted the testing of only one coil, whose performance is to be checked against theory. The coil is large enough, however, to permit rearrangement of the basic computed data into a number of smaller groupings that correspond to smaller coils. This rearrangement greatly multiplies the usefulness of the basic computed data of the large coil, whose specifications are as follows:

4 layers,

70 turns per layer,

0.1285 inch wire diameter (#8AWG),

1 1/8, 1 3/8, 1 5/8, 1 7/8 inches radius of respective layers,

1/4 inch pitch,

Copper wire with rubber-fiber insulation.

Shown below is a list of 35 coils that have been chosen from all that may be derived from the above specifications.

Table 2. Coil Specifications for This Investigation

Inside diameter (inches)	Number of turns per layer	Number of layers
2	4-70*	1
2	4-70*	2
2	4-70*	3
2	4-70*	4
2-1/2	4-70*	1
3	4-70*	1
3-1/2	4-70*	1

*4, 8, 16, 32, 70 turns per layer

The first 20 coils to be studied differ in two respects, namely, number of turns per layer and number of layers. The last 15 coils also differ in two respects, namely, number of turns per layer and the inside diameter. Thus, it is believed that this selected list of coils is both adequate to show the trend of proximity effects in coils and highly efficient in the utilization of the data computed. The schedule of computations is given in Appendix B. Because of their huge volume the numerical tables developed during the computations are omitted from this thesis except for summary data. The computations were performed by the International Business Machine Service Unit of Iowa State College under the direction of Mr. C. C. Mosier, Manager.

Interpretation of Results

Figures 3 - 9 are plots of the data compiled in Tables 3 - 9 from the IBM calculations. Also included are curves for the case of the isolated conductor and for coils having an infinite number of turns. The data for these latter curves were computed from formulas and auxiliary tables published by Arnold.[14] A study of the graphs reveals no startling differences from the comparable work of Butterworth and Arnold. In fact, the continuity of information contained in these curves indicates that the calculations are free from arithmetical blunders. Remarkable agreement between the data for an infinite number of turns and those based on the theoretical development outlined in this thesis is further evidence of the accuracy of the latter work. It will be noticed that each of Arnold's curves in Figures 3 - 6 as well as each curve in Figures 7 - 9 has a point of inflection. The reason for this occurrence is that the special function $G(x)$, which is identified with a uniform field, has a point of inflection near the value $x = 2\frac{1}{4}$, beyond which the slope of $G(x)$ approaches $\sqrt{2}/8$ as a limit. This asymptotic value of the slope is just half that for the function $F(x)$, which is identified with skin effect. These facts also explain the stronger showing of this peculiar property in the curves for the multi-layer coils, for the mean square field is part of the coefficient of $G(x)$ in the expression for the resistance ratio.

Two important checks on the agreement between the development in this thesis and those of Butterworth and Arnold may be made. Use of Arnold's formulas and auxiliary tables for the single-layer coil having 70 turns and $2\frac{1}{4}$ inches diameter yields a resistance ratio of 3.10 at 20,000 cycles per second. In Table 6 is found for this coil the resistance ratio of 3.22, which differs by 3.87 per cent from Arnold's value. Among the many extreme cases developed by Butterworth was the ratio of resistance in a coil to that of the same wire in straightened form at "infinite frequency". This ratio is a fictitious value, being the limit of the same ratio for a finite frequency as the frequency mathematically approaches infinity, without regard to the true physical picture. This ratio may be obtained from the results of this thesis by dividing Equation 19 by the expression for the resistance ratio of the coil winding when straightened and carrying alternating current of the same frequency. This latter expression in Butterworth's notation is

$$\frac{R'}{R_{d-c}} = 1 + F(x). \qquad (20)$$

Table 3. Resistance Ratios for 3 3/4-Inch Diameter Single-Layer Coils

Cps	n / x	4	8	16	32	70	∞ (Arnold)
20,000	4.939	2.395	2.569	2.778	3.026	3.275	3.318*
10,000	3.492	1.739	1.855	1.995	2.161	2.326	2.377
5,000	2.469	1.313	1.380	1.461	1.557	1.653	1.712
2,000	1.5618	1.070	1.089	1.111	1.138	1.164	1.194
1,000	1.1042	1.019	1.024	1.030	1.038	1.045	1.055
500	0.7809	1.005	1.006	1.008	1.010	1.012	1.014
200	0.4939	1.001	1.001	1.001	1.002	1.002	1.002
100	0.3492	1.000	1.000	1.000	1.000	1.000	1.001

*Values in this column were computed from formulas and data published by A. H. M. Arnold.

Table 4. Resistance Ratios for 3 1/4-Inch Diameter Single-Layer Coils

Cps	x \ n	4	8	16	32	70	∞ (Arnold)
20,000	4.939	2.396	2.577	2.794	3.040	3.270	3.292*
10,000	3.492	1.740	1.860	2.005	2.170	2.323	2.357
5,000	2.469	1.313	1.383	1.467	1.562	1.651	1.700
2,000	1.5618	1.070	1.090	1.113	1.139	1.164	1.190
1,000	1.1042	1.019	1.024	1.031	1.038	1.045	1.054
500	0.7809	1.005	1.006	1.008	1.010	1.012	1.014
200	0.4939	1.001	1.001	1.001	1.002	1.002	1.002
100	0.3492	1.000	1.000	1.000	1.000	1.000	1.001

*Values in this column were computed from formulas and data published by A. H. M. Arnold.

Table 5. Resistance Ratios for 2 3/4-Inch Diameter Single-Layer Coils

Cps	x \ n	4	8	16	32	70	∞ (Arnold)
20,000	4.939	2.397	2.584	2.808	3.048	3.254	3.256*
10,000	3.492	1.740	1.865	2.015	2.175	2.313	2.332
5,000	2.469	1.314	1.386	1.473	1.565	1.645	1.684
2,000	1.5618	1.070	1.090	1.114	1.140	1.162	1.185
1,000	1.1042	1.019	1.025	1.031	1.038	1.045	1.052
500	0.7809	1.005	1.006	1.008	1.010	1.011	1.014
200	0.4939	1.001	1.001	1.001	1.002	1.002	1.002
100	0.3492	1.000	1.000	1.000	1.000	1.000	1.001

*Values in this column were computed from formulas and data published by A. H. M. Arnold.

Table 6. Resistance Ratios for 2 1/4-Inch Diameter Single-Layer Coils

Cps	n / x	4	8	16	32	70	∞ (Arnold)
20,000	4.939	2.397	2.591	2.820	3.044	3.221	3.211*
10,000	3.492	1.740	1.870	2.022	2.172	2.290	2.301
5,000	2.469	1.314	1.389	1.477	1.564	1.632	1.665
2,000	1.5618	1.070	1.091	1.116	1.140	1.158	1.179
1,000	1.1042	1.019	1.025	1.032	1.038	1.044	1.051
500	0.7809	1.005	1.006	1.008	1.010	1.011	1.013
200	0.4939	1.001	1.001	1.001	1.002	1.002	1.002
100	0.3492	1.000	1.000	1.000	1.000	1.000	1.001

*Values in this column were computed from formulas and data published by A. H. M. Arnold.

Table 7. Resistance Ratios for 2 1/4-Inch Diameter Two-Layer Coils

Cps	x \ n	4	8	16	32	70
20,000	4.939	4.103	5.420	6.792	8.010	8.943
10,000	3.492	2.880	3.759	4.676	5.489	6.112
5,000	2.469	1.973	2.481	3.011	3.482	3.842
2,000	1.5618	1.253	1.393	1.540	1.670	1.770
1,000	1.1042	1.070	1.109	1.151	1.187	1.215
500	0.7809	1.018	1.028	1.039	1.048	1.055
200	0.4939	1.003	1.005	1.006	1.008	1.009
100	0.3492	1.001	1.001	1.002	1.002	1.002

Table 8. Resistance Ratios for 2 1/4-Inch Diameter Three-Layer Coils

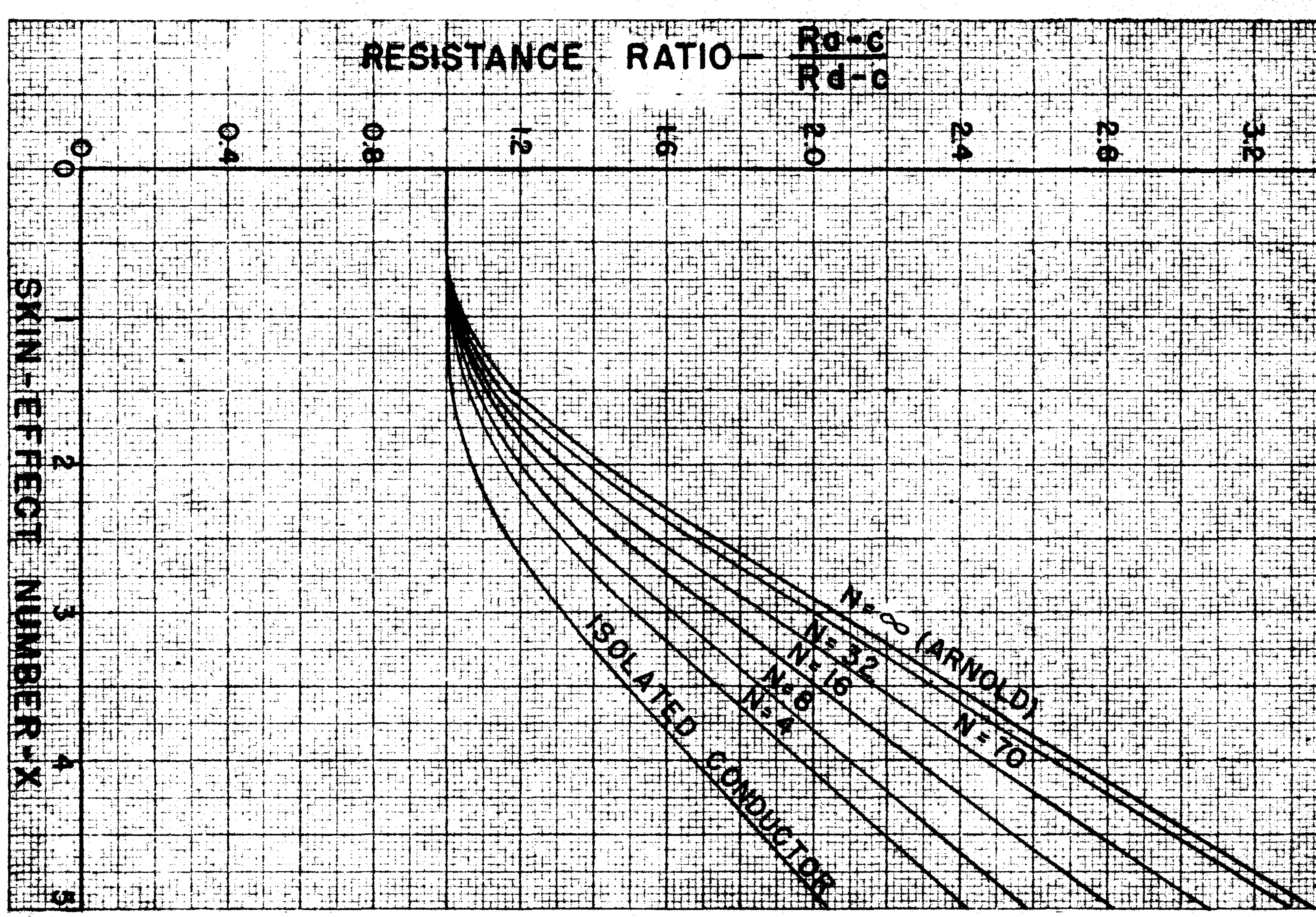
Cps	n / x	4	8	16	32	70
20,000	4.939	5.965	9.100	12.509	15.551	17.915
10,000	3.492	4.124	6.217	8.494	10.526	12.105
5,000	2.469	2.692	3.903	5.219	6.394	7.307
2,000	1.5618	1.452	1.786	2.151	2.476	2.728
1,000	1.1042	1.126	1.220	1.322	1.413	1.484
500	0.7809	1.032	1.057	1.083	1.106	1.125
200	0.4939	1.005	1.009	1.013	1.017	1.020
100	0.3492	1.001	1.002	1.003	1.004	1.005

Table 9. Resistance Ratios for 2 1/4-Inch Diameter Four-Layer Coils

Cps	n / x	4	8	16	32	70
20,000	4.939	7.775	13.180	19.425	25.135	29.672
10,000	3.492	5.332	8.943	13.114	16.928	19.957
5,000	2.469	3.391	5.478	7.890	10.096	11.848
2,000	1.5618	1.645	2.222	2.889	3.499	3.984
1,000	1.1042	1.180	1.342	1.529	1.700	1.836
500	0.7809	1.046	1.088	1.136	1.180	1.216
200	0.4939	1.007	1.014	1.022	1.029	1.035
100	0.3492	1.002	1.004	1.006	1.007	1.009

Figure 3. Resistance Ratio for Single-Layer Coils.

$$\frac{\text{Coil Diameter}}{\text{Wire Diameter}} = 29.18$$



RESISTANCE RATIO — $\frac{R_{a-c}}{R_{d-c}}$
0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
SKIN-EFFECT NUMBER-X
0
1
2
3
4
5
N=∞ (ARNOLD)
N=70
N=32
N=16
N=8
N=4
ISOLATED CONDUCTOR

Figure 4. Resistance Ratio for Single-Layer Coils.

$$\frac{\text{Coil Diameter}}{\text{Wire Diameter}} = 25.29$$

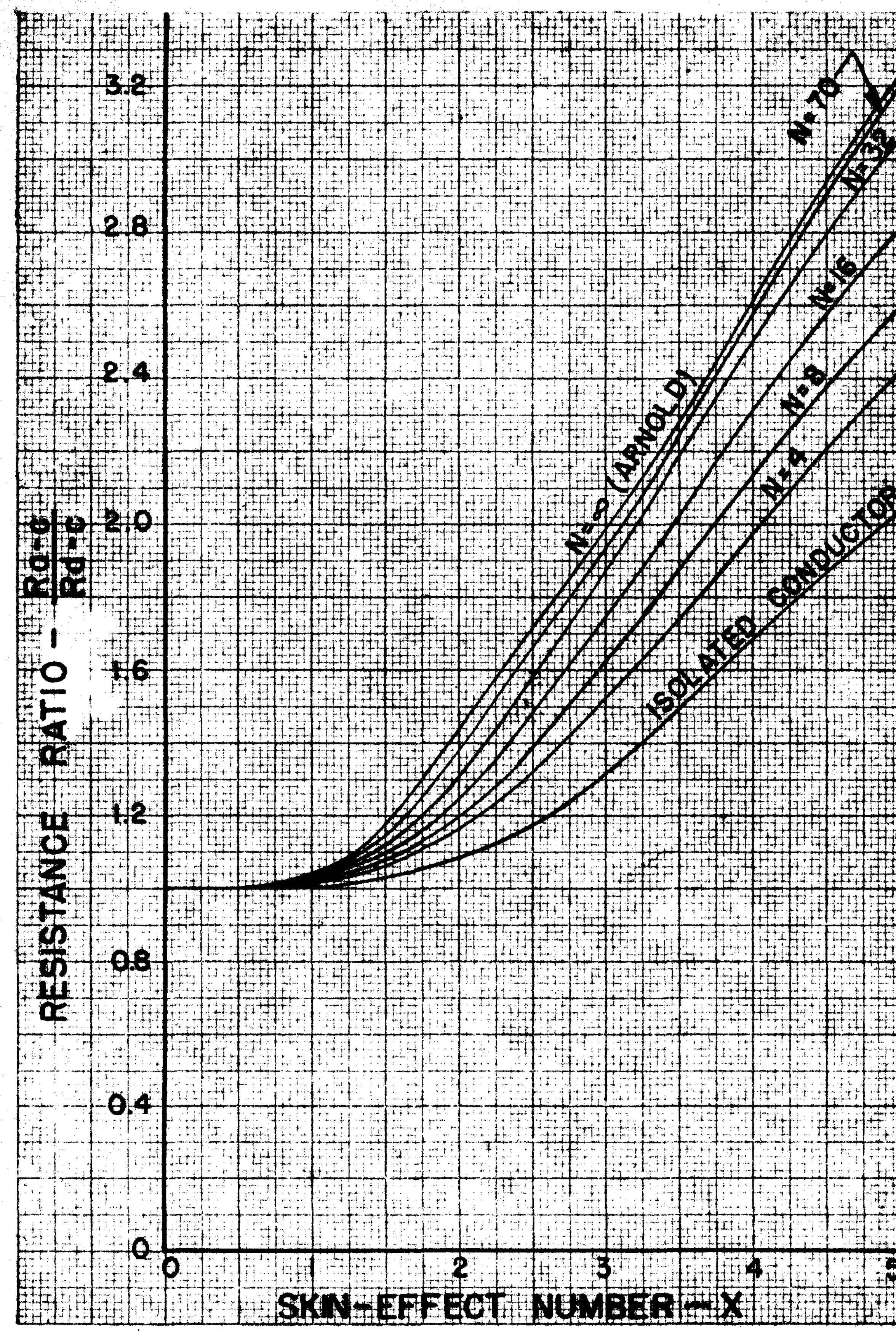
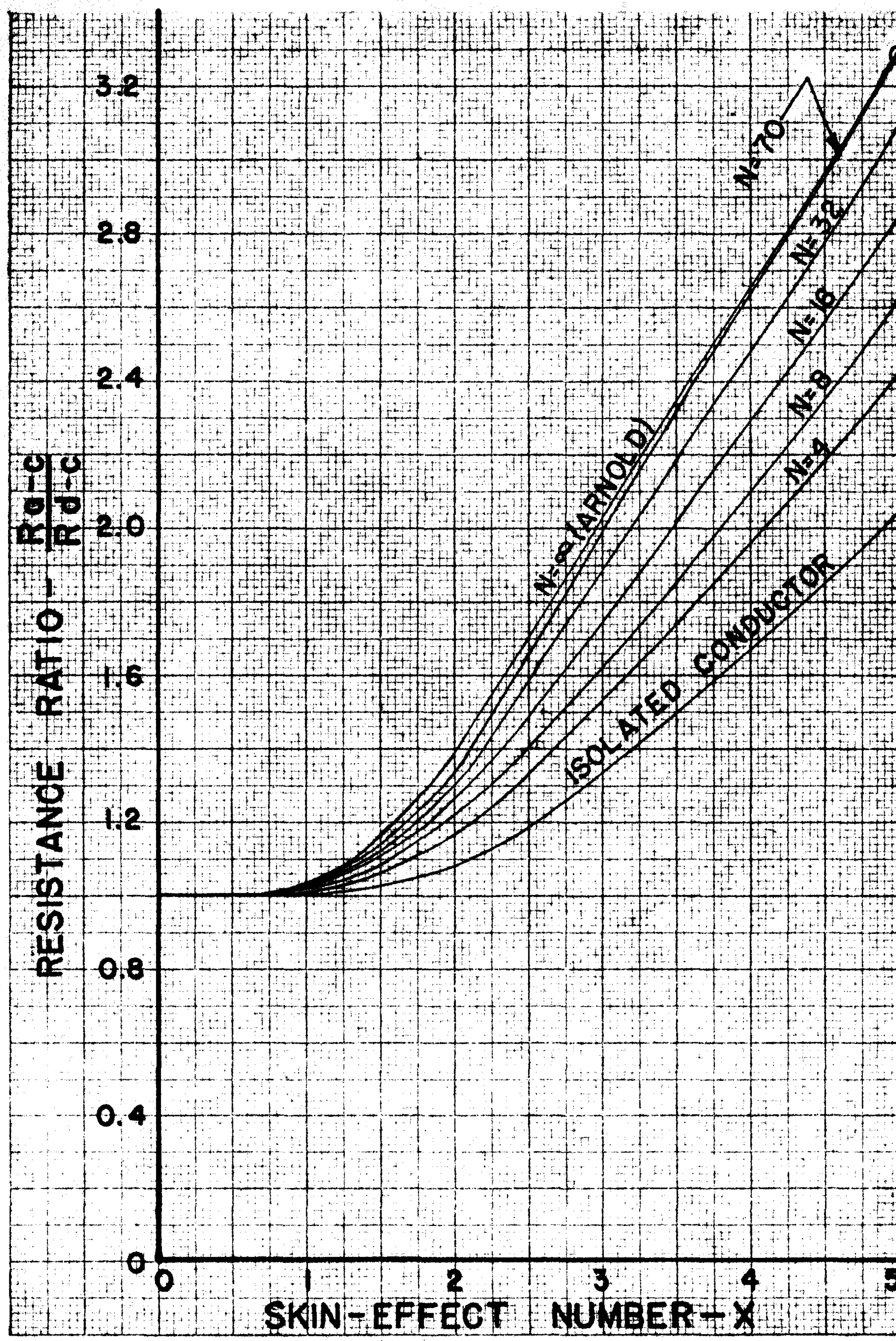

RESISTANCE RATIO — $\frac{Ra\text{-}c}{Rd\text{-}c}$
SKIN-EFFECT NUMBER — X
N=70
N=32
N=16
N=8
N=4
N=∞ (ARNOLD)
ISOLATED CONDUCTOR
0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
0
2
3
4
5

Figure 5. Resistance Ratio for Single-Layer Coils.

$$\frac{\text{Coil Diameter}}{\text{Wire Diameter}} = 21.4$$



RESISTANCE RATIO - Ra-c / Rd-c
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0
0
1
2
3
4
5
SKIN-EFFECT NUMBER - X
N=70
N=32
N=16
N=8
N=4
N=∞ (ARNOLD)
ISOLATED CONDUCTOR

Figure 6. Resistance Ratio for Single-Layer Coils.

$$\frac{\text{Coil Diameter}}{\text{Wire Diameter}} = 17.50$$

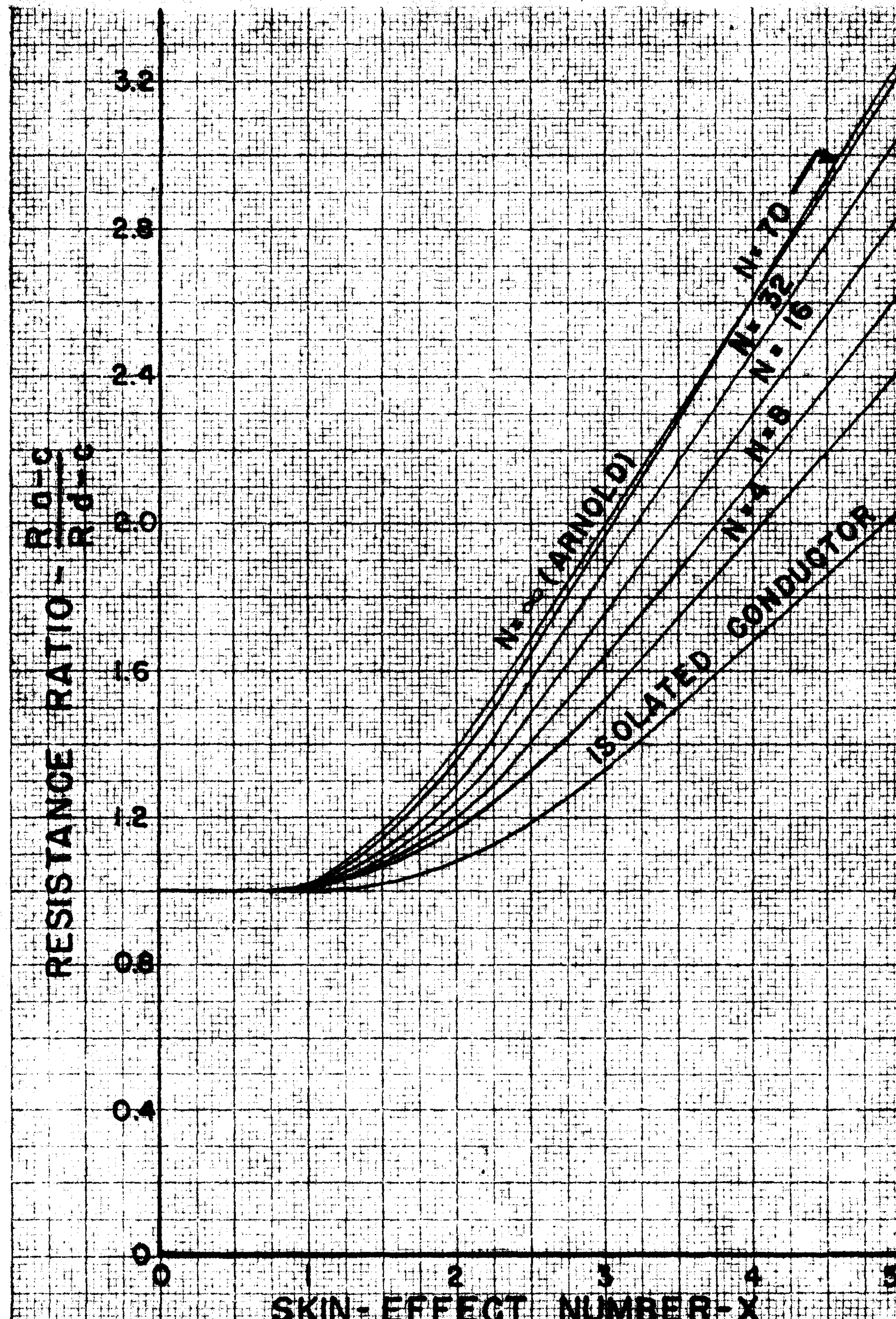
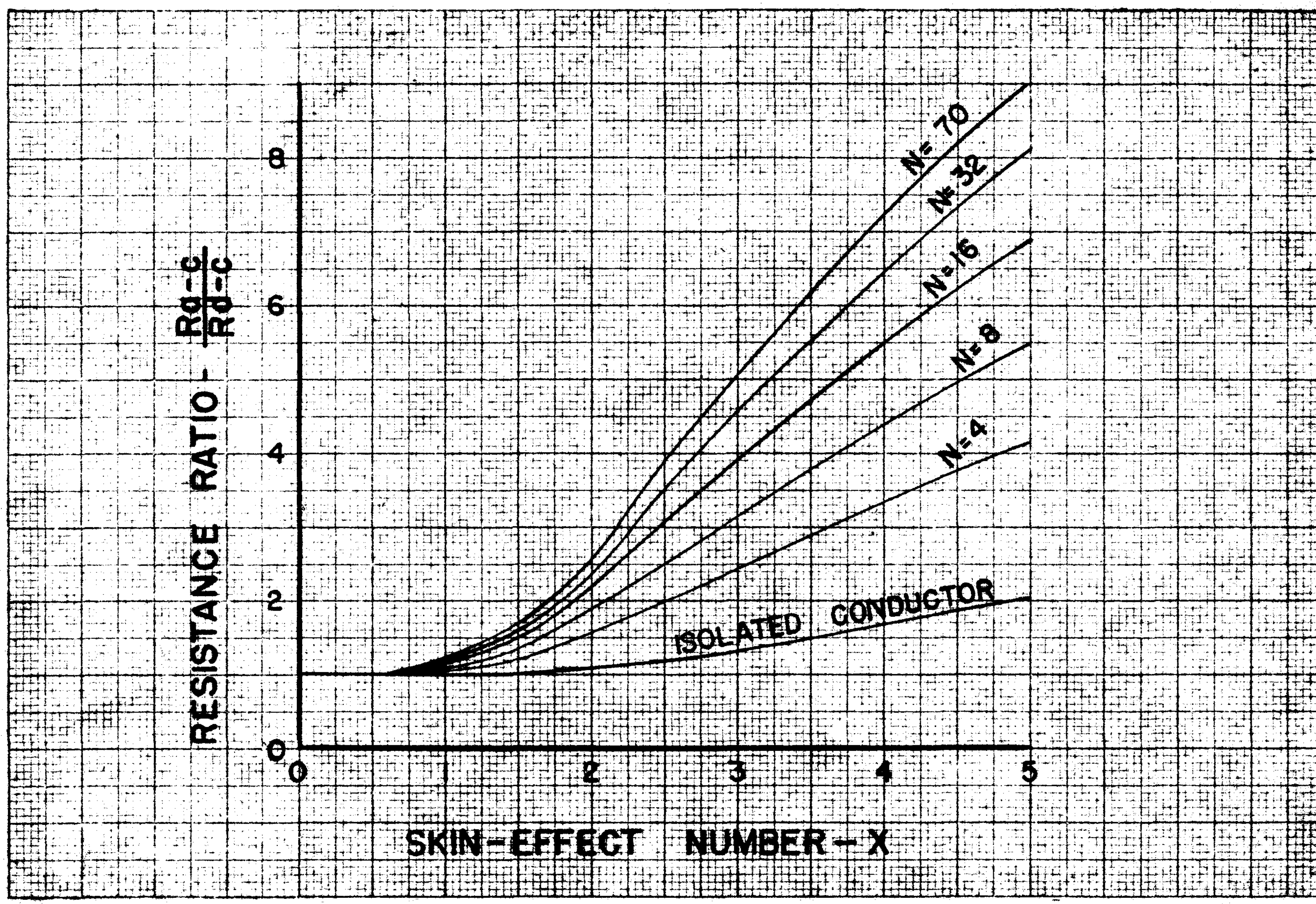
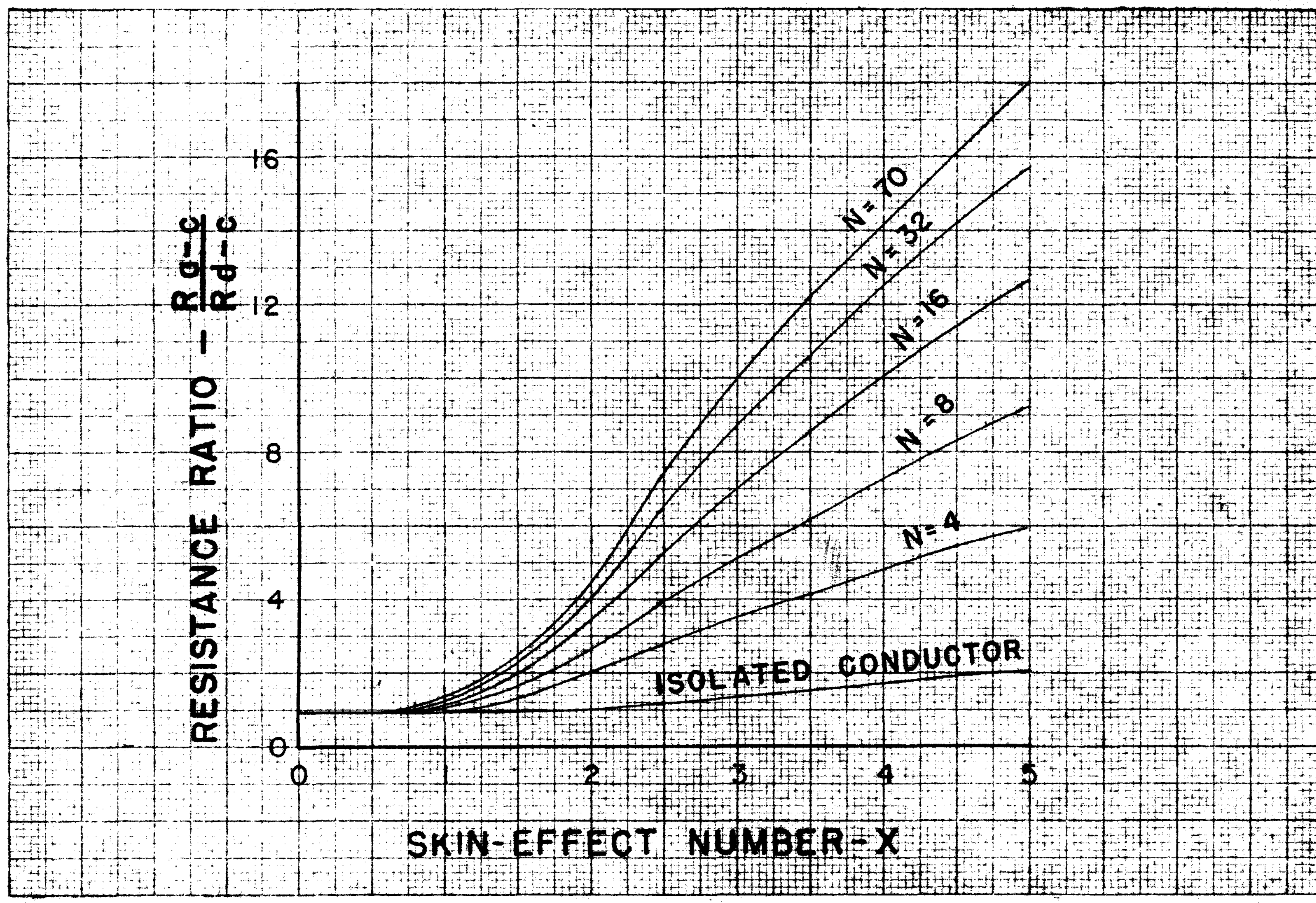

RESISTANCE RATIO - R a-c / R d-c
SKIN-EFFECT NUMBER-X
0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
1
2
3
4
5
N=∞ (ARNOLD)
N=70
N=32
N=16
N=8
N=4
ISOLATED CONDUCTOR

Figure 7. Resistance Ratio for 2-Layer Coils.



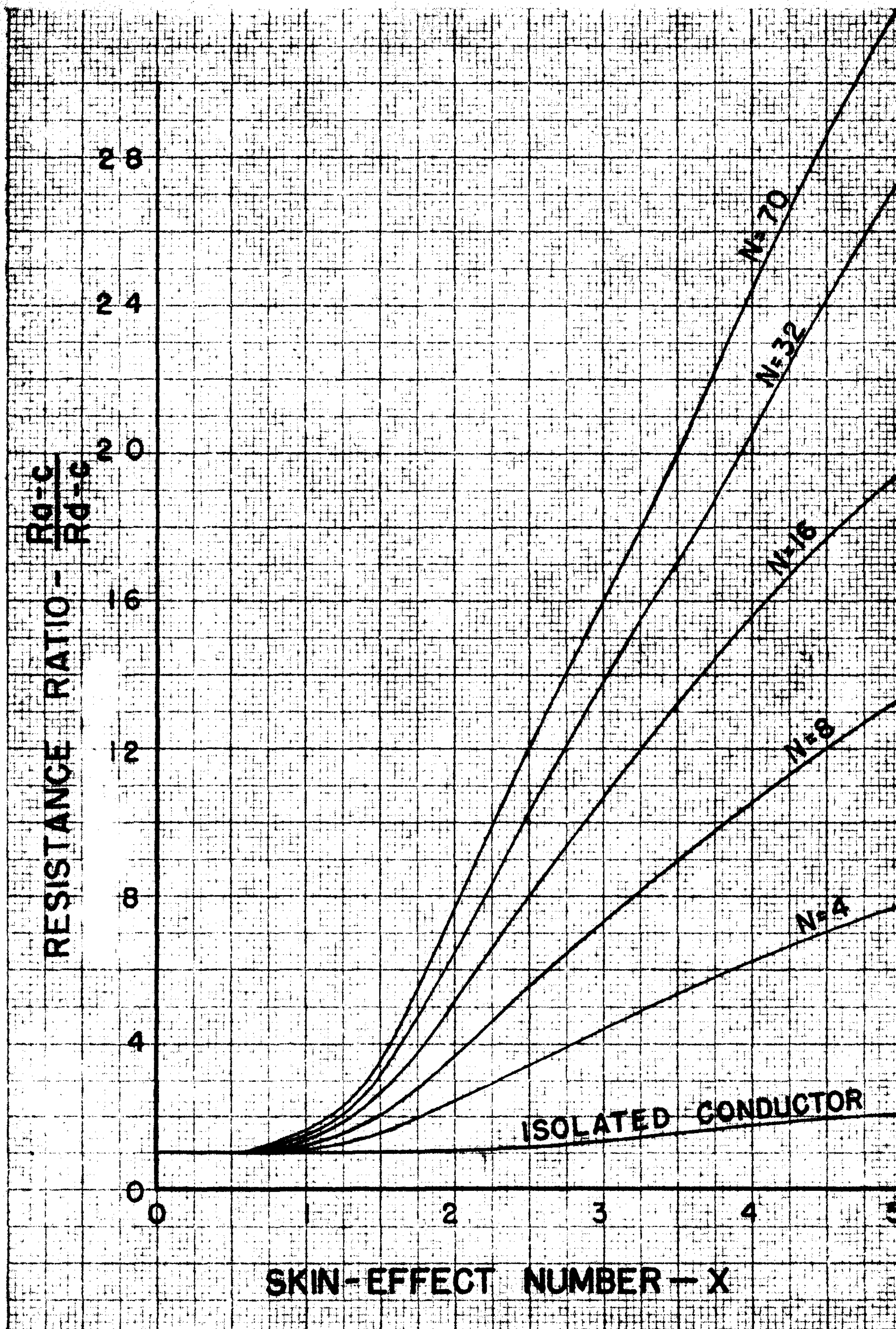
RESISTANCE RATIO - Ra-c / Rd-c
SKIN-EFFECT NUMBER - X
N = 70
N = 32
N = 16
N = 8
N = 4
ISOLATED CONDUCTOR
0
2
4
6
8
0
1
2
3
4
5

Figure 8. Resistance Ratio for 3-Layer Coils.



RESISTANCE RATIO − $\frac{R_{a-c}}{R_{d-c}}$
16
12
8
4
0
0
2
3
4
5
SKIN-EFFECT NUMBER-X
N=70
N=32
N=16
N=8
N=4
ISOLATED CONDUCTOR

Figure 9. Resistance Ratio for 4-Layer Coils.


28
24
20
16
12
8
4
0
0
1
2
3
4
5
RESISTANCE RATIO - Ra-c / Rd-c
SKIN-EFFECT NUMBER — X
N=70
N=32
N=16
N=8
N=4
ISOLATED CONDUCTOR

Performing this division and taking the limit yields the following equation as desired:

$$\frac{R_{a-c}}{R'} = 1 + \tfrac{1}{2}r^2 \frac{\sum_{j=1}^{m}\left[b_j \sum_{i=1}^{n}\left(\frac{4\pi}{I} H_{ij}\right)^2\right]}{n\sum_{j=1}^{m} b_j}. \qquad (21)$$

Although the functions F(x) and G(x) approach infinity as a limit as the frequency approaches infinity, the ratio of G(x) to F(x) approaches $\frac{1}{2}$ as a limit.

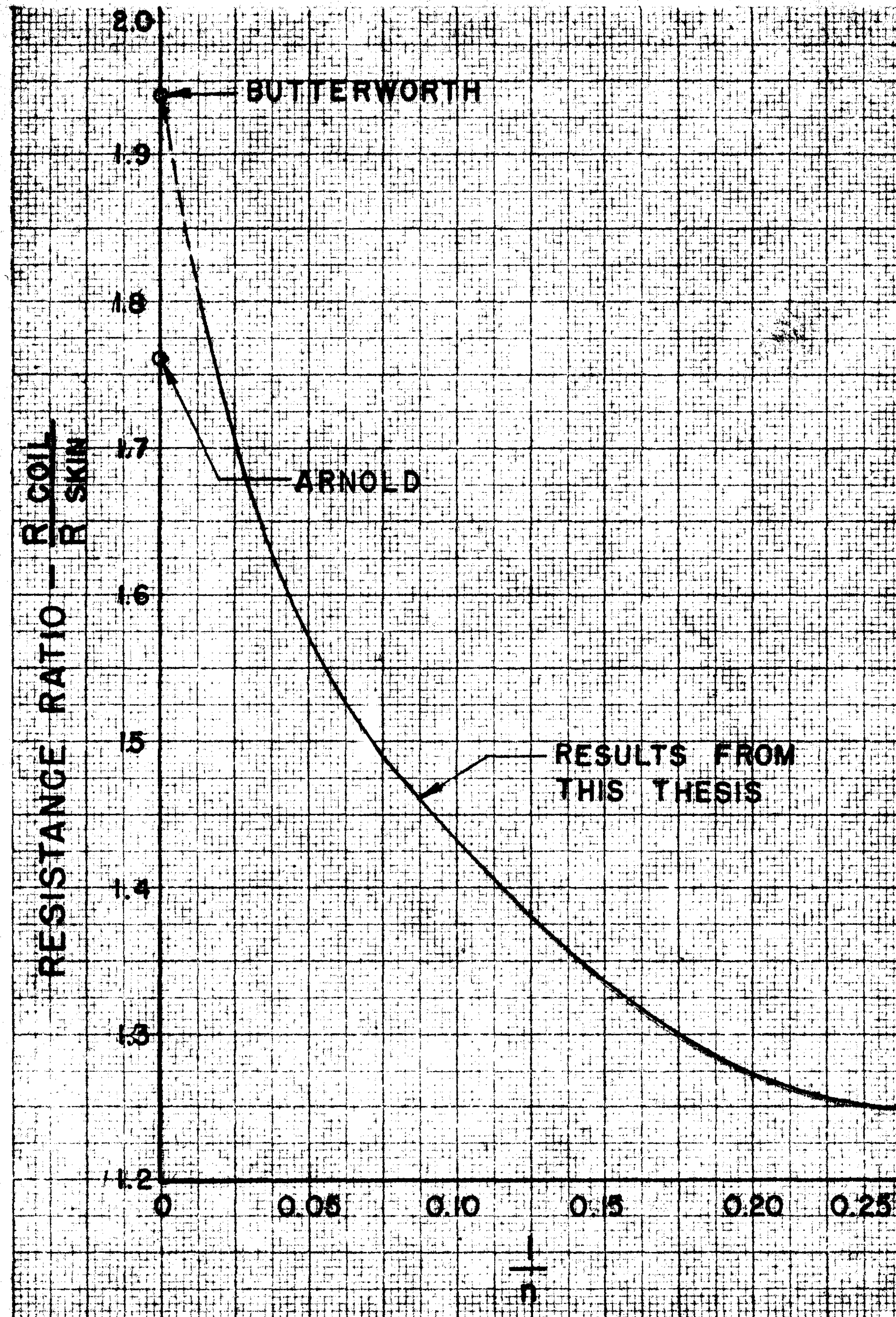
Since Butterworth's "infinite frequency" data[11] are available only for single-layer coils of finite length but having an infinite number of turns, it was found necessary to plot the resistance ratios obtained by Equation 21 against the reciprocal of the number of turns for a group of single-layer coils having the same diameter. The $2\frac{1}{4}$-inch diameter coils were chosen, and the results are shown in Table 10 and Figure 10. The coils used in this thesis have a wire-diameter-to- pitch ratio of 0.514. The corresponding interpolated value of resistance ratio R_{a-c}/R' from Butterworth's Table 6, page 703 (Reference 11) is 1.97, which is plotted in Figure 10. It can be seen that this point forms a continuous extension of the curve plotted from the data in Table 10. It is interesting to note, however, that the same point computed from Arnold's formulas and data is 1.76, which is unexpectedly low. This discrepancy is believed attributable to the fact that Arnold's formulas and tables were based on Butterworth's developments for coils with an

Table 10. "Infinite - Frequency" Values for 2¼-Inch Diameter Single-Layer Coils

Number of turns n	$\frac{1}{n}$	$\frac{R_{a-c}\text{ (coil)}}{R'\text{ (skin-effect)}}$
4	0.25	1.252
8	0.125	1.383
16	0.0625	1.536
32	0.03125	1.687
70	0.01429	1.806

infinite number of turns, Arnold having modified these developments by semi-empirical means for coils with a finite number of turns. Thus, theoretical confirmation from the best known work of others seems to exist for the theoretical developments proposed in this thesis. There remains to be shown what confirmation has been found in experimental studies.

Figure 10. "Infinite-Frequency" Characteristics of $2\frac{1}{4}$-Inch Diameter Single-Layer Coils.


2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
0
0.05
0.10
0.15
0.20
0.25
BUTTERWORTH
ARNOLD
RESULTS FROM THIS THESIS
RESISTANCE RATIO — R COIL / R SKIN
1 / n

EXPERIMENTAL APPROACH

Choice of Test Equipment

As indicated earlier, the coils chosen for this thesis were wound with heavy copper in order to emphasize the phenomena of skin and proximity effects in the audio-frequency range. This choice, however, resulted in coils of extremely low resistance. A thorough search revealed no presently available equipment designed for this particular range of vector impedance. A study of many bridge circuits indicated that the chief limiting factors for the most promising circuits were availability of sufficiently accurate standard impedances and the requirement of an extremely large amount of tuning capacity.

Consequently, it was a fortunate coincidence that there was found available a model of the Western Electric 4 - A Impedance Bridge. Figure 11 shows the basic circuit, which is known as a hybrid-coil type bridge. Because this circuit also requires an extremely large amount of tuning capacity, it was necessary to abandon the idea of testing all but the largest coil wound for this investigation. This bridge, however, fulfills the requirements of adequately calibrated standard impedances. As shown schematically in Figure 11, the circuit consists of a balanced transformer with two center-tapped secondaries. The primary (not shown) is excited by an oscillator. The standard and unknown impedances are connected as shown and are easily interchanged by means of a

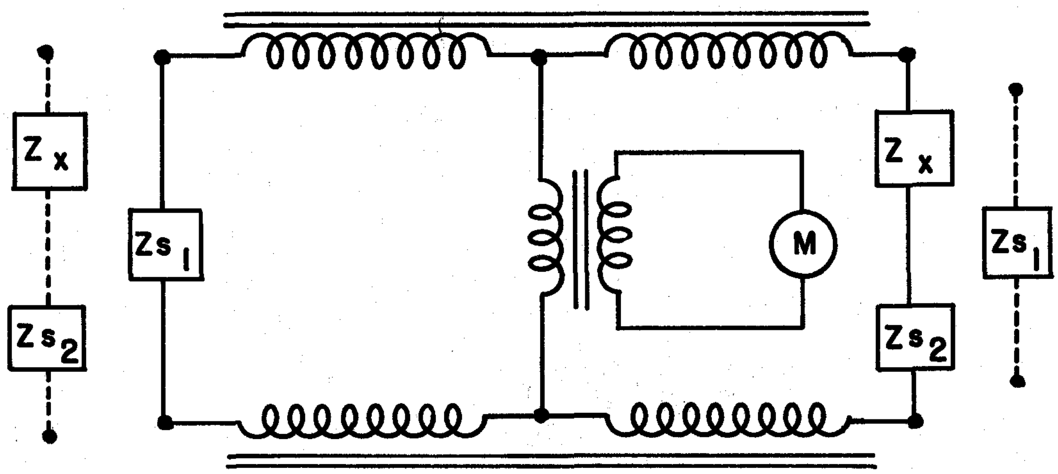


Figure 11. Hybrid-Coil Type Bridge.

switching arrangement provided in the equipment. Two calibrated decades, one resistance and the other capacitance, are also included in the equipment. The switching arrangement provides many circuit configurations, thus making the bridge very versatile. The following combinations are available:

Z_{s1}	Z_{s2}
R and C	0
R	C
R	0
C	0

The second combination was used for this investigation. In addition, provision for isolating Z_x and each of the decades for external use is made. Finally, the bridge is completely shielded.

The principle of operation of the bridge is easily understood by comparing it to a three-wire single-phase system that is fed by balanced generators at both ends. The detector through an isolating transformer measures the unbalanced potential in the "neutral wire". Since the center-tapped secondaries are balanced, any unbalance will be due to unequal impedances. Actually, the bridge is sufficiently well balanced to meet all ordinary requirements, but for this application it was necessary to consider the errors due to any unbalance.

Test Procedure

In Figure 12 is shown a working diagram of the hybrid-coil bridge. A survey of the circuit revealed only negligible unbalance in the electromotive forces provided by the center-tapped secondaries. The impedances as found in the equipment are indicated on the diagram. The resistance decade was recalibrated. In order to permit a finer adjustment in the bridge a calibrated decade resistance box was connected in parallel with the resistance decade already in the bridge. With these data it was possible to correct the readings obtained when testing the coil.

In addition to these efforts to minimize errors, an analysis was made of the effect of unavoidable errors in the hybrid-coil bridge.

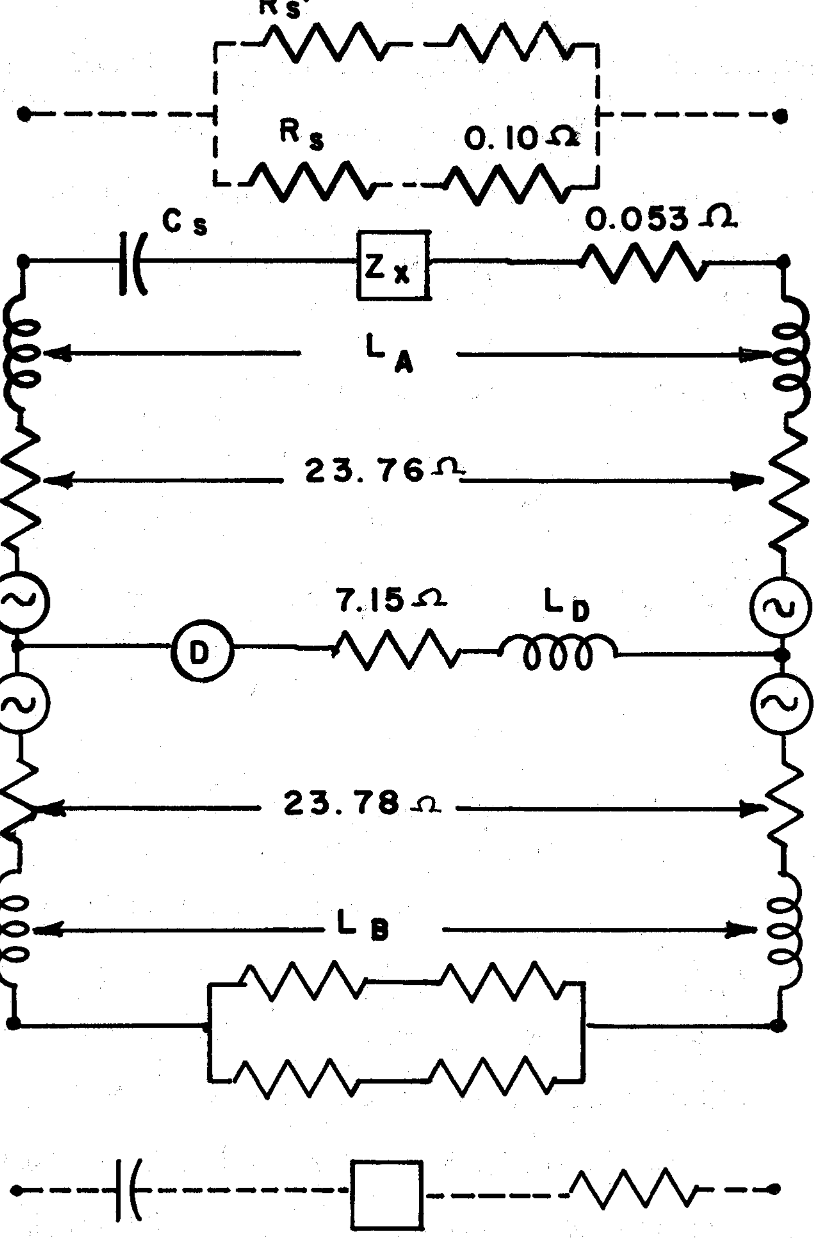


Figure 12. Impedance Survey of Bridge.

Consider the two halves of the bridge as A and B with impedance Z_a and electromotive force E_a associated with the A half and similar values Z_B and E_B with the B half. Let the unknown impedance Z_x be balanced in the A half by a standard impedance Z_s' in the B half. Then it follows that

$$\frac{E_A}{Z_x + Z_A} = \frac{E_B}{Z_s' + Z_B} ; \tag{22}$$

whence,

$$Z_s' = \frac{E_B}{E_A}(Z_x + Z_A) - Z_B . \tag{23}$$

If E_A and E_B are assumed equal, an estimate of Z_x would be

$$Z_x' = Z_s' + Z_B - Z_A . \tag{24}$$

Substitution for Z_s' yields the equation

$$Z_x' = \frac{E_B}{E_A} Z_x + \frac{E_B - E_A}{E_A} Z_A . \tag{25}$$

If the unknown impedance Z_x is balanced in the B half by a standard impedance Z_s'' in the A half, similar relationships to those above are obtained, and a new estimate of Z_x is

$$Z_x'' = \frac{E_A}{E_B} Z_x + \frac{E_A - E_B}{E_B} Z_B . \tag{26}$$

The arithmetic mean of these two estimates is

$$\frac{Z_x' + Z''_x}{2} = \frac{E_A^2 + E_B^2}{2E_AE_B} Z_x + \frac{E_B - E_A}{2E_AE_B}(E_BZ_A - E_AZ_B), \tag{27}$$

while the geometric mean is

$$\sqrt{Z_x' Z_x''} = \sqrt{Z_x^2 + Z_x\left(\frac{E_A - E_B}{E_A E_B}\right)(E_B Z_B - E_A Z_A) - \left(\frac{E_A - E_B}{E_A E_B}\right)^2 E_A E_B Z_A Z_B}. \quad (28)$$

Since the errors may be assumed as small, the above equation may be closely approximated as

$$\sqrt{Z_x' Z_x''} \cong Z_x + \frac{E_A - E_B}{2E_A E_B}\left(E_B Z_B - E_A Z_A\right). \quad (29)$$

Further study shows that on the average a better estimate of Z_x is formed by taking the arithmetic mean of the two means already obtained than would be had by either the arithmetic mean or the geometric mean alone. This practice was followed in the processing of the original data. In addition, each original reading was repeated twice, yielding three estimates of Z_x for each of the two bridge arrangements. The arithmetic mean of these estimates was taken and used as data for the process described above. These original estimates differed from each other by 3.2 per cent or less, the rms deviation being 1.23 per cent. It is believed that the final determinations of the coil resistance at the different frequencies tested are as free from error as is possible within the limitations of the equipment used. As stated earlier, the coil tested has an inductance of 590 microhenrys and a distributed capacitance of 257.5 micromicrofarads.* Therefore, the error due to

*These values were determined by means of a Boonton Q-Meter between 250 and 370 kc.

distributed capacity is less than 0.5 per cent.

Interpretation of Results

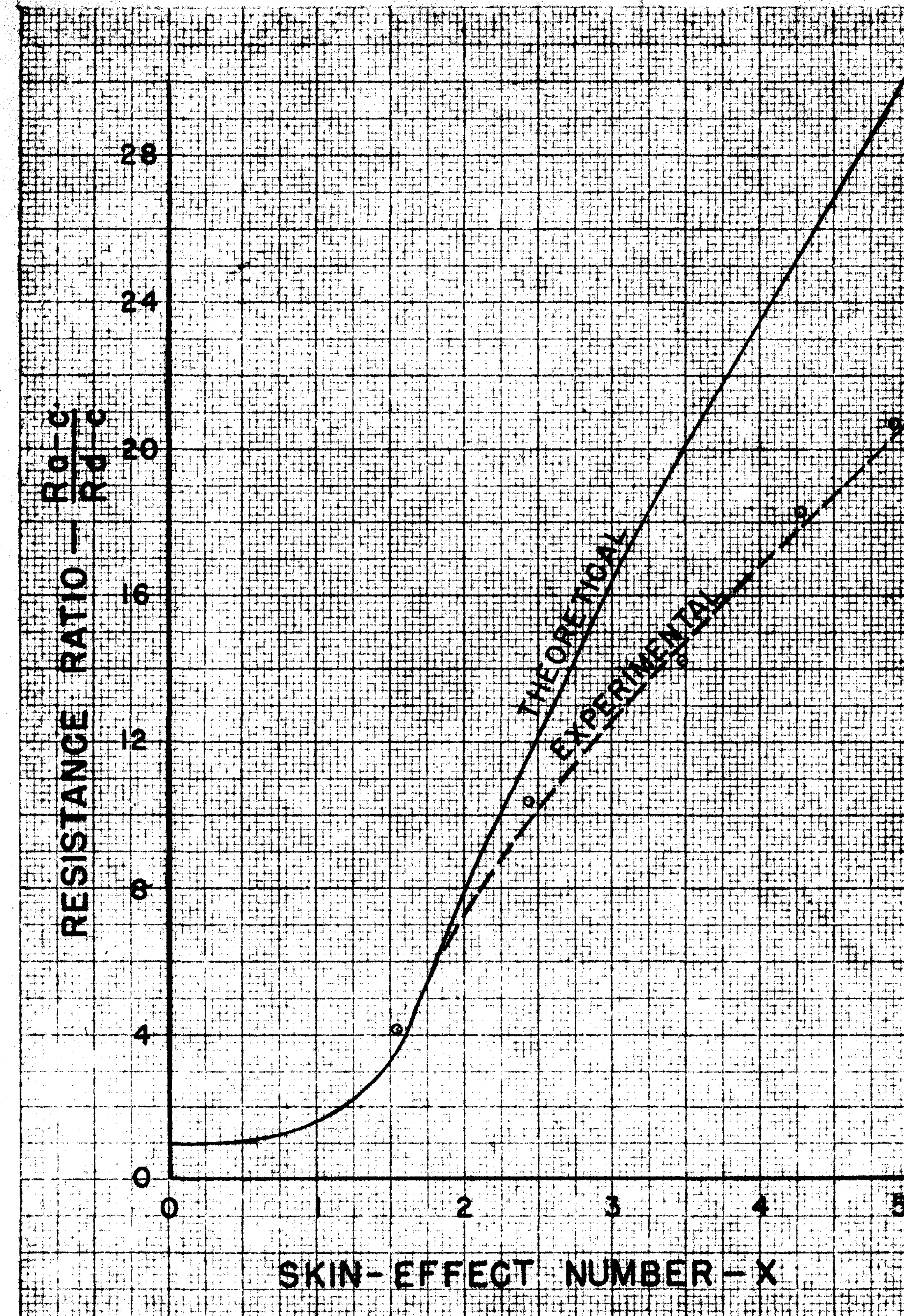
Table 11 shows the experimental results, which are also plotted

Table 11. Variation of Resistance and Inductance with Frequency

Frequency	Skin-Effect Number	Coil Resistance	Resistance Ratio	Coil Inductance
kc	x	ohms	R_{d-c}=0.145	μh
20	4.939	3.005	20.7	628
15	4.277	2.659	18.33	643
10	3.492	2.076	14.31	635
5	2.469	1.496	10.32	648
2	1.562	0.592	4.08	683

in Figure 13. The theoretical curve is taken from Figure 9 for comparison. It can be seen that there is considerable departure between the experimental and theoretical curves, becoming 32 per cent of the theoretical value at X = 5. There is also a discrepancy between the inductance determination by means of the Boonton Q-meter and the data

Figure 13. Resistance Ratio for 4-Layer Coil of 70 Turns per Layer.


RESISTANCE RATIO — $\frac{Ra-c}{Rd-c}$
28
24
20
16
12
8
4
0
0
1
2
3
4
5
SKIN-EFFECT NUMBER — X
THEORETICAL
EXPERIMENTAL

in Table 11. A simple calculation shows the internal inductance of the wire in the coil to be 3.5 microhenrys. The total inductance of the coil cannot change by more than this amount throughout the range of frequencies employed in this thesis. Therefore, these discrepancies even within Table 11 cannot be explained by change in internal inductance alone.

The data for Table 11 were gathered after experience with the use of the hybrid-coil bridge had been gained, but comparison with data taken a week earlier shows a remarkable similarity. See Table 12.

Table 12. Comparison of Old and New Inductance Determinations

Frequency	Old	New
kc	μh	μh
20	629	628
15	641	643
10	635	635
5	649	648
2	682	683

Close agreement between the two sets of values, the elapsed time of one week between their accumulation, and the fact that each old value was a mean of ten observations while each new value was the mean of six observations, would hardly support the contention that the peculiar variations in Table 12 were a coincidence. Efforts have been made to determine the possible causes of error in the experimental arrangement. Although the capacitance decade had been checked for leakage and found satisfactory, a mathematical check was made on the data for such an effect but none was found. The audio oscillator had been calibrated against the available 10-kc standard and related frequencies. The remaining unknowns are the capacitance decade calibration and the stray impedances in the test circuit. The calibration of the capacitance decade is believed to be sufficiently satisfactory to preclude the capacitance readings as a source of error. Hence, it seems that both stray impedances and stray energy couplings must account for the experimental differences from theory. Although these stray phenomena have not been fully identified, they were consistently present throughout the tests and affected those portions of the circuit that were switched when the coil and the standard impedances were interchanged. Therefore, most of the discrepancy between the two curves in Figure 13 appears to be due to unknown experimental errors.

SUMMARY

The determination of coil resistance as a function of frequency is an extremely difficult problem whether viewed analytically or experimentally. As a theoretical problem the analysis of the electrical properties of a coil cannot be accomplished satisfactorily without a thorough understanding of fundamental electromagnetic principles. Such an understanding makes clear that Maxwell's field relations are the basis for the more commonly known circuit concepts and relations when suitable approximations are applied. Therefore, the strictly geometrical definitions of resistance and inductance often found in texts are inapplicable to the problem of this thesis. Instead, it is necessary as has been demonstrated earlier to begin with Maxwell's field relations and to apply suitable approximations with full cognizance of their consequences on the validity of the results.

While many individuals have attempted the analysis of electrical coils, of those individuals whose work has been described in the literature all except Butterworth have failed to meet the requirement of applying suitable approximations with full cognizance of their consequences on the validity of the results. Butterworth's work, however, fell short of meeting the needs of coil designers, because Butterworth limited his developments to relatively simple but extreme situations. It remained for Arnold to extend Butterworth's work to include the more complex but more useful coil designs but only for single-layer coils.

Arnold accomplished this extension by semi-empirical means. The efforts described in this thesis were toward the goal of finding suitable formulas for predicting the resistance ratio of multilayer coils in the audio-frequency range.

It was found that well below the natural frequency, coil losses may be roughly divided into four groups, namely, (1) losses based on uniform current distribution, (2) skin-effect losses, (3) those due to a uniform alternating magnetic field, and (4) losses due to a field that is not uniform. Butterworth[11] has shown that the total loss in a coil is the sum of these components. For coils that are not too closely wound the effect of field distortion may be neglected as was done in this thesis.

The use of IBM equipment solved the dilemma of choosing between a fruitless search for simplified estimates of the mean square field in a multilayer coil and the laborious job of accurately computing this value. A large selection of coils was made, and their specifications were punched on IBM cards. Instructions based on exact (to a high order of approximation) formulas for magnetic field strength were also punched into IBM cards. The contribution to the field on the wire axis of any turn by all the other turns was computed and the results summed vectorially. The mean square field of the entire coil was obtained by summing the squares of the field strength of each turn. This information was used for calculating the resistance ratio as a function of frequency on the basis of the formula developed in this thesis.

These resistance-ratio data were compared to the work of Butterworth and Arnold and were found to be in good agreement. Experimental confirmation is incomplete for lack of adequate facilities to measure the extremely low resistances of the chosen coils. The one experimental curve obtained was for the four-layer coil of 70 turns per layer, and this curve departed from the theoretical curve by as much as 32 per cent for a value of 5 for the skin-effect number. The explanation for this discrepancy between theory and experiment is to be found both in the unsolved problem of field distortion and in the need for a satisfactory means of determining the resistance component of a high-reactance low-resistance coil. The sound theoretical foundation underlying the developments in this thesis as well as close agreement with the theoretical work of others provides the basis for believing that a substantial contribution has been made to an understanding of the subject.

LITERATURE CITED

1. Maxwell, James Clerk. A treatise on electricity and magnetism. Second edition. Volumes 1 and 2. London, Oxford at the Clarendon Press. 1881.

2. Carson, J. R. Electromagnetic theory and the foundations of electric circuit theory. Bell System Technical Journal. 6:1-17. 1927.

3. King, Ronald W. P. Electromagnetic engineering. Volume 1. New York, McGraw-Hill Book Company, Inc. 1945.

4. Ramo, Simon, and Whinnery, John R. Fields and waves in modern radio. New York, John Wiley and Sons, Inc. 1944.

5. Sommerfeld, A. Über das Wechselfeld und den Wechelstromwiderstand von Spulen und Rollen. Annalen der Physik. Fourth series, 15:673-708. 1904.

6. Coffin, J. G. The influence of frequency upon the self-inductance of cylindrical coils of m-layers. The Physical Review. 23:193-211. 1906.

7. Cohen, Louis. The influence of frequency on the resistance and inductance of solenoidal coils. Bulletin of the Bureau of Standards. 4, no. 1:161-178. 1907.

8. Howe, G. W. O. The application of telephone transmission formulae to skin-effect problems. The Journal of the Institution of Electrical Engineers. 54:473-480. 1916.

9. __________. The high-frequency resistance of wires and coils. The Journal of the Institution of Electrical Engineers. 58:152-162, Discussion 162-170. 1920.

10. Butterworth, S. Eddy-current losses in cylindrical conductors, with special applications to the alternating current resistances of short coils. Philosophical Transactions of the Royal Society of London. Series A, 222:57-100. 1922.

11. ________. On the alternating current resistance of solenoidal coils. Proceedings of the Royal Society of London. Series A, 107:693-715. 1925.

12. ________. Effective resistance of inductance coils at radio frequency. Experimental Wireless and the Wireless Engineer. 3:203-210, 309-316, 417-424, 483-492, Discussion 579-580. 1926.

13. Hickman, C. N. Alternating-current resistance and inductance of single-layer coils. Bureau of Standards Science Paper. No. 472. 19:73-104. 1923.

14. Arnold, A. H. M. The resistance of round-wire single-layer inductance coils. The Proceedings of the Institution of Electrical Engineers. Monograph No. 9, 98, Part 4:94-100. 1951.

15. Rosa, E. B. and Grover, F. W. Formulas and tables for the calculation of mutual and self-inductance (revised). Scientific Papers of the Bureau of Standards. No. 169. Third Edition. 1916.

16. Arnold, A. H. M. Proximity effect in solid and hollow round conductors. The Journal of the Institution of Electrical Engineers. 88, Part 2:349-359. 1941.

17. Medhurst, R. G. H.f. resistance and self-capacitance of single-layer solenoids. Experimental Wireless and the Wireless Engineer. 24:35-43, 80-92, Discussion 185, 281. 1947.

18. McLachlan, N. W. Bessel functions for engineers. London, Oxford University Press. 1934.

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APPENDICES

APPENDIX A

Eddy-Current Losses in a Long Conducting Cylinder in an Alternating Magnetic Field

The following analysis closely follows that made by Butterworth.[10] Since the skin-effect relations (Equations 5 - 7) will be used in what follows, all the basic assumptions underlying these equations will apply to this analysis. The particular situation to be investigated is the eddy-current loss in an infinitely long nonmagnetic right circular cylinder of uniform electrical conductivity in a magnetic field perpendicular to the axis of the cylinder and distributed in a manner that is independent of any distance measured along the axis. Although this case is an ideal one, it is very useful because an exact analysis is possible and many actual problems in skin and proximity effects can either be treated exactly or approximated with considerably reduced work by application of the results of this analysis.

In Figure 14 is shown a portion of an infinitely long cylinder and the cylindrical coordinates to be used. The axis of the cylinder is chosen as the Z - axis and an arbitrary point O on the axis is taken to be the origin. The perpendicular distance between an arbitrary point P in space and the Z - axis is r. If P' is the projection of P on a reference plane normal to the Z - axis and passing through the

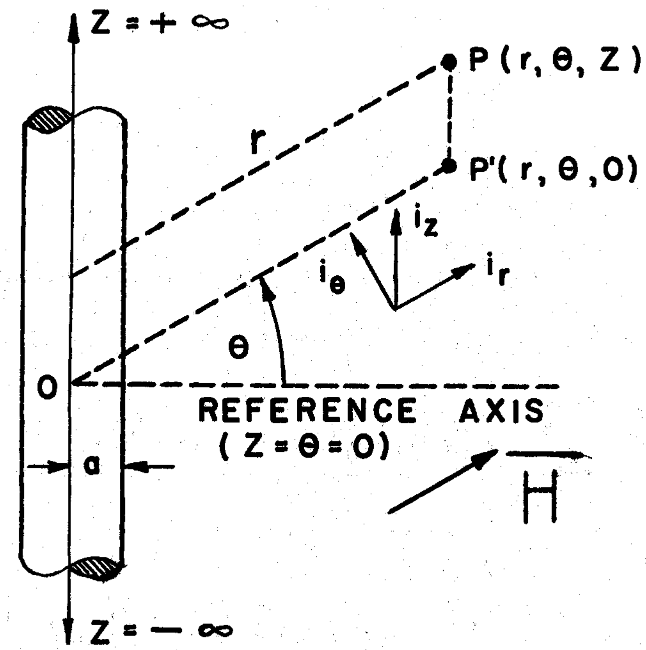

Figure 14. Infinitely Long Conducting Cylinder in a Magnetic Field.

origin, the angle between the line $\overline{OP'}$ and an arbitrary reference axis through the origin and in the reference plane is designated as θ. Thus any point in space is completely identified by the quantities r, θ, and Z. In accordance with the statement of the problem the magnetic field is represented as

$$\overline{H} = i_r H_r + i_\theta H_\theta \; . \tag{30}$$

The symbols i_r, i_θ, and i_z are three mutually perpendicular unit vectors forming a right-hand system. Each unit vector extends in the direction of maximum increase of the variable indicated in the subscript.

Equations 5a and 30, when combined, show the vector $\overline{E}$ to consist of only one component, which is in the axial direction and defined by the relationship

$$\sigma E_z = \frac{\partial H_\theta}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \theta} + \frac{H_\theta}{r} . \qquad (31)$$

Expansion of Equation 5b with the knowledge that $E_r = E_\theta = 0$, yields the following equations:

$$\frac{1}{r} \frac{\partial E_z}{\partial \theta} = -\mu \frac{\partial H_r}{\partial t} , \qquad (32a)$$

$$\frac{\partial E_z}{\partial r} = \mu \frac{\partial H_\theta}{\partial t} . \qquad (32b)$$

Elimination of H_r and H_θ from Equations 31 and 32 leads to the fundamental relationship for any time-wise variation of E_z:

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} = \mu \sigma \frac{\partial E_z}{\partial t} . \qquad (33)$$

Since E_z is assumed to be a sinusoidal function of time, Equation 33 is changed to read

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} = j \omega \mu \sigma E_z . \qquad (34)$$

A solution of Equation 34 is

$$E_z = E' \cos n\theta + E'' \sin n \theta , \qquad (35)$$

where E' and E" are functions of r only. Substitution of Equation 35 into Equation 34 yields two equations, one of which is

$$\frac{\partial^2 E'}{\partial r^2} + \frac{1}{r}\frac{\partial E'}{\partial r} - \left(\frac{n^2}{r^2} + j\,\omega\mu\sigma\right)E' = 0 \qquad (36)$$

and the second of which is the same with E" substituted for E'. These are Bessel equations of the n^{th} order and have the general solution.

$$E' = A_n' J_n(\lambda r) + T_n' Y_n(\lambda r), \qquad (37)$$

where J_n and Y_n are n^{th} order Bessel functions of the first and second kinds, respectively, of the argument λr, and A_n' and T_n' are arbitrary constants. The quantity λ is defined by

$$\lambda^2 = -j\,\omega\mu\sigma = -j\left(\frac{x}{r}\right)^2 \qquad (38)$$

and is thus related to the skin-effect number x defined in Equation 3. Since E' is finite at all points in space but Y_n approaches infinity as r approaches zero, it follows that T_n' is zero.

Because σ is finite inside the cylinder and zero outside, the solution

$$E' = A_n' J_n(\lambda r) \qquad (37')$$

is only for inside the cylinder. Outside the cylinder Equation 36 becomes

$$\frac{\partial^2 E'}{\partial r^2} + \frac{1}{r}\frac{\partial E'}{\partial r} - \frac{n^2}{r^2} E' = 0 , \qquad (36')$$

which has the solutions

$$E' = B_o' \ln r + C_o' \quad (n = 0) , \qquad (39a)$$

$$E' = B_n' r^n + C_n' r^{-n} \; (n \neq 0) . \qquad (39b)$$

The quantities E_z and $\frac{\partial E_z}{\partial r}$ are continuous at the boundary of the cylinder. Therefore, the solutions of Equations 36 and 36' must agree at the boundary. Accordingly, it follows that

$$A_o' J_o (\lambda a) = B_o' \ln a + C_o' \quad , \qquad (40a)$$

$$A_n' J_n (\lambda a) = B_n' a^n + C_n' a^{-n} \quad (n \neq 0), \qquad (40b)$$

$$\lambda A_o' J_o' (\lambda a) = B_o'/a , \qquad (40c)$$

$$\lambda A_n' \left[- \frac{n}{\lambda a} J_n (\lambda a) + J_{n-1} (\lambda a) \right]$$

$$= nB_n' a^{n-1} - nC_n' a^{-n-1} \; (n \neq 0), \qquad (40d)$$

where a is the radius of the cylinder. These four simultaneous equations permit the reduction of the number of arbitrary constants from six to two. For later convenience the constants B_o' and B_n' are arbitrarily retained, and it follows that

$$A_o' = \frac{B_o'}{\lambda a J_o' (\lambda a)} , \qquad (41a)$$

$$C_o' = \frac{B_o' J_o(\lambda a)}{\lambda a J_o'(\lambda a)} - B_o' \ln a , \qquad (41b)$$

$$A_n' = \frac{2n B_n' a^{n-1}}{\lambda J_{n-1}(\lambda a)} \quad (n \neq 0) , \qquad (41c)$$

$$C_n' = a^{2n} B_n' \frac{J_{n+1}(\lambda a)}{J_{n-1}(\lambda a)} \quad (n \neq 0) . \qquad (41d)$$

Equations 36 -41, inclusive, hold for E" as well as for E' except that double primes replace single primes.

A more general solution of Equation 34 is a Fourier series composed of terms as shown in Equation 35. Accordingly, it follows that inside the cylinder

$$E_z(\text{inside}) = \frac{B_o J_o(\lambda r)}{\lambda a J_o'(\lambda a)} + \frac{2}{\lambda a} \sum_{n=1}^{\infty} \frac{n B_n a^n J_n(\lambda r)}{J_{n-1}(\lambda a)} \cos(n\theta + \alpha_n) \qquad (42)$$

and outside the cylinder

$$E_z(\text{outside}) = B_o \left[\frac{J_o(\lambda a)}{\lambda a J_o'(\lambda a)} + \ln \frac{r}{a} \right] + \sum_{n=1}^{\infty} B_n \left[r^n + a^n \left(\frac{a}{r}\right)^n \frac{J_{n+1}(\lambda a)}{J_{n-1}(\lambda a)} \right] \cos(n\theta + \alpha_n), \qquad (43)$$

where

$$B_n = \sqrt{B_n'^2 + B_n''^2} \qquad (44)$$

and

$$\alpha_n = -\tan^{-1} \frac{B_n''}{B_n'} \,. \qquad (45)$$

Equations 32a and 32b for sinusoidal time variations become

$$H_r = j \frac{1}{\omega\mu r} \frac{\partial E_z}{\partial \theta} \,, \qquad (46a)$$

$$H_\theta = -j \frac{1}{\omega\mu} \frac{\partial E_z}{\partial r} \,. \qquad (46b)$$

Substitution of the solutions for E_z into these last equations yields the solutions for the magnetic field. Accordingly, inside the cylinder

$$H_r(\text{inside}) = -j \frac{2}{\omega\mu\lambda\, ar} \sum_{n=1}^{\infty} \frac{n^2 B_n a^n J_n(\lambda r)}{J_{n-1}(\lambda a)} \sin(n\theta + \alpha_n) \qquad (47)$$

and

$$H_\theta(\text{inside}) = -j \frac{1}{\omega\mu\, a} \left[\frac{B_o J_o'(\lambda r)}{J_o'(\lambda a)} + 2 \sum_{n=1}^{\infty} \frac{n B_n a^n J_n'(\lambda r)}{J_{n-1}(\lambda a)} \cdot \cos(n\theta + \alpha_n) \right] , \qquad (48)$$

and outside the cylinder

$$H_r \text{ (outside)} = -j \frac{1}{\omega\mu r} \sum_{n=1}^{\infty} B_n n \left[r^n + a^n \left(\frac{a}{r}\right)^n \frac{J_{n+1}(\lambda a)}{J_{n-1}(\lambda a)} \right] \cdot \sin(n\theta + \alpha_n) \qquad (49)$$

and

$$H_\theta \text{(Outside)} = -j \frac{1}{\omega\mu} \left\{ \frac{B_o}{r} + \sum_{n=1}^{\infty} B_n n \left[r^{n-1} - \frac{a^{2n}}{r^{n+1}} \frac{J_{n+1}(\lambda a)}{J_{n-1}(\lambda a)} \right] \cdot \cos(n\theta + \alpha_n) \right\}. \qquad (50)$$

These expressions for the electric and magnetic fields show E_z, H_r, and H_θ to be continuous at the surface of the cylinder as required by the boundary conditions and agree with the results obtained by Butterworth.[10]

By Poynting's theorem, the instantaneous rate p of energy flow into the cylinder is

$$p = - \int_s \overline{E} \times \overline{H} \cdot d\overline{S} \ . \qquad (51)$$

Since $\overline{E} = i_z E_z$, $\overline{H} = i_r H_r + i_\theta H_\theta$, and $d\overline{S} = i_r a l d\theta$ for a length l of the cylinder, it follows that

$$p = - \int_s (i_z E_z) \times (i_r H_r + i_\theta H_\theta) \cdot (i_r\, a l d\theta) \qquad (52a)$$

$$= -al \int_0^{2\pi} (i_\theta E_z H_r - i_r E_z H_\theta) \cdot (i_r d\theta) \qquad (52b)$$

$$= al \int_0^{2\pi} E_z H_\theta d\theta \quad \text{watts,} \tag{52c}$$

where E_z and H_θ are instantaneous quantities evaluated at the surface of the cylinder. For time-averaged power P_{av} due to sinusoidally varying fields Equation 52c leads to the expression

$$P_{av} = \frac{al}{4} \int_0^{2\pi} (E_z \tilde{H}_\theta + \tilde{E}_z H_\theta) d\theta \quad \text{watts,} \tag{53}$$

where E_z and H_θ are now complex quantities and $\tilde{E}_z$ and $\tilde{H}_\theta$ the conjugates of E_z and H_θ, respectively, all evaluated at the surface of the cylinder. When $r = a$ in the expressions for E_z and H_θ (both inside and outside), the following expressions are obtained:

$$E_z = \frac{B_o J_o(\lambda a)}{\lambda a\, J_o'(\lambda a)} + \frac{2}{\lambda a} \sum_{n=1}^{\infty} \frac{n B_n a^n J_n(\lambda a)}{J_{n-1}(\lambda a)} \cos(n\theta + \alpha_n), \tag{54}$$

$$H_o = -j \frac{1}{\omega \mu a} \left[B_o + 2 \sum_{n=1}^{\infty} \frac{n B_n a^n J_n'(\lambda a)}{J_{n-1}(\lambda a)} \cos(n\theta + \alpha_n) \right]. \tag{55}$$

It is convenient at this point to introduce new symbols for the sake of both brevity and clarity. Let

$$\chi_o = \frac{J_o(\lambda a)}{\lambda a J_o'(\lambda a)} \tag{56}$$

and

$$\chi_n = \frac{J_{n+1}(\lambda a)}{J_{n-1}(\lambda a)} \quad (n \neq 0) \; . \tag{57}$$

Then, the expressions for E_z and H_θ become:

$$E_z = B_o \chi_o + \sum_{n=1}^{\infty} B_n a^n (1 + \chi_n) \cos (n\theta + \alpha_n) \tag{58}$$

$$H_\theta = -j \frac{1}{\omega \mu a} \left[B_o + \sum_{n=1}^{\infty} n B_n a^n (1 - \chi_n) \cos (n\theta + \alpha_n) \right] \tag{59}$$

The integrand in Equation 53 may now be expanded and the terms grouped according to the functions of θ. Four groups of terms will be found as follows:

(1) Terms free of θ,

(2) Terms involving $\cos (n\theta + \alpha_n)$ alone,

(3) Terms involving $\cos (m\theta + \alpha_m) \cos (n\theta + \alpha_n)$ $(m \neq n)$,

(4) Terms involving $\cos^2 (n\theta + \alpha_n)$.

The corresponding integrals listed in the same order are as follows:

(1) $$\int_0^{2\pi} d\theta = 2\pi, \tag{60}$$

(2) $$\int_0^{2\pi} \cos (n\theta + \alpha_n) d\theta = 0, \tag{61}$$

(3) $$\int_0^{2\pi} \cos (m\theta + \alpha_m) \cos (n\theta + \alpha_n) d\theta = 0 \quad (m \neq n), \tag{62}$$

$$(4) \quad \int_0^{2\pi} \cos^2(n\theta + \alpha_n) = \pi. \tag{63}$$

Substitution of Equations 58 and 59 into Equation 53 and utilization of the integrals just listed yields the following expression for average power per unit length of the cylinder:

$$\frac{P_{av}}{l} = \frac{\pi}{2\,\omega\mu}\left\{B_o\tilde{B}_o\left[j\,(\chi_o - \tilde{\chi}_o)\right] + \sum_{n=1}^{\infty} nB_n\tilde{B}_n a^{2n}\left[j(\chi_n - \tilde{\chi}_n)\right]\right\}. \tag{64}$$

In order to make this new expression more useful, several changes of variables will be made. First the B's, which are not convenient for either use or interpretation, will be replaced by their equivalents in terms of the Fourier coefficients of the undistorted field. The expressions for the undistorted field are obtained from Equations 47 - 50 by letting σ and, consequently, λ approach zero as a limit. Thus, the following expressions are obtained:

$$H_r \text{ (inside) (undistorted)} = -j\frac{1}{\omega\mu}\sum_{n=1}^{\infty} B_n n r^{n-1} \sin(n\theta + \alpha_n), \tag{65}$$

$$H_\theta \text{ (inside) (undistorted)} = -j\,\frac{1}{\omega\mu}\left[\frac{B_o r}{a^2} + \sum_{n=1}^{\infty} B_n n r^{n-1} \cdot \cos(n\theta + \alpha_n)\right], \tag{66}$$

$$H_r \text{ (outside) (undistorted)} = -j \frac{1}{\omega\mu} \sum_{n=1}^{\infty} B_n n r^{n-1} \sin (n\theta + \alpha_n), \tag{67}$$

$$H_\theta \text{ (outside) (undistorted)} = -j \frac{1}{\omega\mu} \left[\frac{B_o}{r} + \sum_{n=1}^{\infty} B_n n r^{n-1} \cdot \cos (n\theta + \alpha_n) \right] . \tag{68}$$

It is interesting to observe that the expressions for the undistorted magnetic field both inside and outside the conductor differ only in the B_o terms. These two terms represent both the internal and external fields resulting from a uniform current density in the conductor it- self. Consequently, the quantity B_o is related to the properties of an isolated conductor carrying current.

If the conversions

$$B_o = j\, \omega\mu\, K_o \tag{69}$$

and

$$B_n = j\, \omega\mu \frac{K_n}{n} \tag{70}$$

are made, Equations 65 - 68 become

$$H_r \text{ (inside) (undistorted)} = \sum_{n=1}^{\infty} K_n r^{n-1} \sin (n\theta + \alpha_n), \tag{71}$$

$$H_\theta \text{ (inside) (undistorted)} = \frac{K_o r}{a^2} + \sum_{n=1}^{\infty} K_n r^{n-1} \cos (n\theta + \alpha_n), \tag{72}$$

$$H_r \text{ (outside) (undistorted)} = \sum_{n=1}^{\infty} K_n r^{n-1} \sin (n\theta + \alpha_n), \tag{73}$$

$$H_\theta \text{ (outside) (undistorted)} = \frac{K_o}{r} + \sum_{n=1}^{\infty} K_n r^{n-1} \cos (n\theta + \alpha_n). \tag{74}$$

These last expressions for the undistorted magnetic field are the most convenient for a Fourier representation.

Next, let

$$\chi_n = \phi_n - j\psi_n . \tag{75}$$

then,

$$\tilde{\chi}_n = \phi_n + j\psi_n , \tag{76}$$

and

$$j(\chi_n - \tilde{\chi}_n) = 2\psi_n . \tag{77}$$

Substitution into Equation 64 of the relations indicated by Equations 69 - 70, and 77 yields the following expression for average power per unit length of the cylinder:

$$\frac{P_{av}}{l} = \pi \omega \mu \left(K_o^2 \psi_o + \sum_{n=1}^{\infty} \frac{K_n^2}{n} a^{2n} \psi_n\right) \text{ watts/meter.} \tag{78}$$

This equation is the chief objective of the analysis just completed, for it expresses the eddy-current losses in an infinitely long conducting cylinder (on a per meter length basis) in a sinusoidally varying magnetic field that is both normal to the axis of the cylinder and invariant along this axis. Moreover, this expression for power is in terms of the radius a of the cylinder, the angular frequency ω, the Fourier components $K_{0,1,2,\ldots\ldots}$ of the magnetic field, the magnetic permeability μ of the cylinder, and the functions $\psi_{0,1,2,\ldots\ldots}$ of the skin-effect number x, which is defined by Equation 3. The nature of the functions $\psi_{0,1,2,\ldots\ldots}$ will now be shown by combining Equations 56, 57, and 77 and recalling Equation 38. The following equations are standard definitions as found in texts[18] on the subject (x = X when r = a):

$$J_o(\lambda a) = J_o(j^{3/2} X) = \text{ber } X + j \text{ bei } X , \quad (79)$$

$$J_o'(\lambda a) = j^{-3/2} (\text{ber}' X + j \text{ bei}' X), \quad (80)$$

$$J_n(\lambda a) = \text{ber}_n X + j \text{ bei}_n X . \quad (81)$$

Therefore,

$$\chi_o = \frac{\text{ber } X + j \text{ bei } X}{X(\text{ber}' X + j \text{ bei}' X)} , \quad (82)$$

$$\chi_n = \frac{\text{ber}_{n+1} X + j \text{ bei}_{n+1} X}{\text{ber}_{n-1} X + j \text{ bei}_{n-1} X} , \; (n \neq 0) \quad (83)$$

and $\tilde{\chi}_o$ and $\tilde{\chi}_n$ are likewise represented except that the sign of j is changed. It followsthat

$$\psi_0 = \frac{1}{X}\left[\frac{\text{ber } X \text{ bei}' X - \text{ber}' X \text{ bei } X}{(\text{ber}' X)^2 + (\text{bei}' X)^2}\right] , \tag{84}$$

$$\psi_n = \frac{\text{ber}_{n+1} X \text{ bei}_{n-1} X - \text{ber}_{n-1} X \text{ bei}_{n+1} X}{(\text{ber}_{n-1} X)^2 + (\text{bei}_{n-1} X)^2} . \tag{85}$$

It was stated earlier (see text immediately following Equation 68) that the quantity B_o is related to the properties of an isolated conductor carrying current. Accordingly, the quantities K_o and χ_o are also related to this case. In fact, the external magnetic field of an isolated conductor carrying current is $\frac{I}{2\pi r}$, hence, $K_o = \frac{I}{2\pi}$ and $K_n = 0$. From Equation 78 it is seen that

$$\frac{P_{av}}{l} = \frac{\omega \mu I^2}{4\pi} \psi_o . \tag{86}$$

Pursuance of this example shows that the ratio of resistance to an alternating current to the direct-current resistance is expressed as

$$\frac{R_{a-c}}{R_{d-c}} = \frac{X^2}{2}\psi_o = 1 + F(X) = \frac{X}{2}\frac{W}{Y} . \tag{87}$$

This ratio has been computed and tabulated by a number of authorities; the form $1 + F(X)$ is used by Butterworth[12] and the form $\frac{X}{2}\frac{W}{Y}$ by Rosa and Grover[15].

If the alternating magnetic field of the general analysis is uniform in strength and direction, there is only one component, namely,

$K_1 = H$, where H is the amplitude of the field strength. From Equation 78 it is seen that

$$\frac{P_{av}}{l} = \pi \omega \mu a^2 H^2 \psi_1 \; . \tag{88}$$

Butterworth chose to define a new function

$$G(X) = \frac{X^2}{8} \psi_1 \; , \tag{89}$$

in terms of which Equation 88 is changed to read as

$$\frac{P_{av}}{l} = \frac{8 \pi H^2}{\sigma} G(X). \tag{90}$$

Butterworth[12] has tabulated values of G(X).

These two examples indicate in a small way the possible usefulness of Equation 78. In fact, this usefulness is limited mostly by the tedious work required to evaluate the Fourier coefficients K_i of the alternating magnetic field and the values of the Bessel functions ψ_1. Arnold[16] has enlarged upon the work of Butterworth and has provided expansions and tables of the Bessel functions ϕ_{1-4} and ψ_{1-4} that should prove adequate for most applications.

APPENDIX B

Schedule of Computations for Determination of the Resistance Ratio of Selected Coils

In order to obtain the maximum information from the theory outlined in the text, the following computations were performed by IBM equipment:

(1) Computation and printing of the individual contributions of all the remaining turns to the H_r and H_z of any turn.

(2) Summation of the H_r and H_z contributions algebraically in order to obtain the resultant H_r and H_z, respectively, for each and every turn in the coil. Since the data of (1) were computed for a total of 4 layers and 70 turns per layer, these data were summed in various smaller groups corresponding to various smaller coils.

(3) Summation of the total field squared for each turn; thus,

$$H_{ij}^2 = (\Sigma H_r)^2 + (\Sigma H_z)^2 \,. \tag{91}$$

(4) Summation of the H_{ij}^2 for each coil configuration.

(5) Computation of the ratio $\frac{R_{a-c}}{R_{d-c}}$ for all coil configurations at the selected frequencies.

These calculations were programmed in the following manner:

(1) Calculation of k. The expression for k^2 given earlier is

$$k^2 = \frac{4\,ab}{(a + b)^2 + z^2} \quad . \tag{92}$$

This equation shows plainly that k is symmetrical with respect to a and b. The chosen values of a, b, and z are as follows:

a = 9/8, 11/8, 13/8, 15/8 inches,

b = 9/8, 11/8, 13/8, 15/8 inches,

z = 0, 1/4, 2/4, 3/4, --------, 69/4 inches.

These sets of values are to be used in all possible combinations except as noted below. According to the number of values of the parameters chosen and to the symmetry just noted, there appears to be a total number of 700 values of k to be computed. Since, however, the influence of a turn upon its own field will be excluded (in this case a = b and z = o), the number of computed values of k is reduced to 696. These computed values of k may be tabulated for convenience as shown in Table 13. The quantities j, j', and i' correspond to b, a, and z, respectively, as shown in this table.

Table 13. Suggested Arrangement of Data for Clarity in the Computation of k.

j	b	j'	a	i'	z	k
1	9/8"	1	9/8"	2	1/4"	
				3	2/4	
				4	3/4	
				⋮	⋮	
				70	69/4	
1	9/8	2	11/8	1	0	
				2	1/4	
				3	2/4	
				⋮	⋮	
				70	69/4	
1	9/8	3	13/8	1	0	
				2	1/4	
				3	2/4	
				⋮	⋮	
				70	69/4	
1	9/8	4	15/8	1	0	
				2	1/4	
				3	2/4	
				⋮	⋮	
				70	69/4	
2	11/8	1	9/8	1	0	Same
				2	1/4	as for
				3	2/4	j=1
				⋮	⋮	j'= 2
				70	69/4	
2	11/8	2	11/8	2	1/4	
				3	2/4	
				4	3/4	
				⋮	⋮	
				70	69/4	
2	11/8	3	13/8	1	0	
				2	1/4	
				3	2/4	
				⋮	⋮	
				70	69/4	
2	11/8	4	15/8	1	0	
				2	1/4	
				3	2/4	
				⋮	⋮	
				70	69/4	

Continued on next page.

Table 13 (concluded)

j	b	j'	a	i'	z	k
3	13/8"	1	9/8"	1	0	Same as
				2	1/4	for j=1
				3	2/4	j'= 3
				⋮	⋮	
				70	69/4	
3	13/8	2	11/8	1	0	Same as
				2	1/4	for j=2
				3	2/4	j' = 3
				⋮	⋮	
				70	69/4	
3	13/8	3	13/8	2	1/4	
				3	2/4	
				4	3/4	
				⋮	⋮	
				70	69/4	
3	13/8	4	15/8	1	0	
				2	1/4	
				3	2/4	
				⋮	⋮	
				70	69/4	
4	15/8	1	9/8	1	0	Same as
				2	1/4	for j=1
				3	2/4	j'= 4
				⋮	⋮	
				70	69/4	
4	15/8	2	11/8	1	0	Same as
				2	1/4	for j=2
				3	2/4	j'= 4
				⋮	⋮	
				70	69/4	
4	15/8	3	13/8	1	0	Same as
				2	1/4	for j=3
				3	2/4	j'= 4
				⋮	⋮	
				70	69/4	
4	15/8	4	15/8	2	1/4	
				3	2/4	
				4	3/4	
				⋮	⋮	
				70	69/4	

(2) Calculation of K and E. The quantities K and E are complete elliptic integrals of the first and second kinds, respectively, both to the modulus k. There are a number of tables available, the most extensive tables being those of A. M. Legendre (London, 1934, Cambridge University Press).* The computed values of K and E can be tabulated in columns adjacent to the values of k in Table 13.

(3) Calculation of $\frac{4\pi}{I} H_r$ and $\frac{4\pi}{I} H_z$.** For convenience the ratio $\frac{R_{a-c}}{R_{d-c}}$ is expressed in terms of $\frac{4\pi}{I} H_{ij}$. Therefore, Equations 12 and 13 are rearranged to read as follows:

$$\frac{4\pi}{I} H_r = \frac{1}{b}\sqrt{4 - k^2\left(\frac{a}{b} + 2 + \frac{b}{a}\right)}\left[\frac{2 - k^2}{2(1 - k^2)} E - K\right]\frac{z}{|z|} , \qquad (93)$$

$$\frac{4\pi}{I} H_z = \frac{k}{\sqrt{ab}}\left[K - \frac{2 - k^2\left(\frac{a}{b} + 1\right)}{2(1 - k^2)} E\right] . \qquad (94)$$

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*The tables actually used were in H. B. Dwight's Mathematical Tables, McGraw-Hill Book Company, New York, 1941, pages 199-205.

**Although these equations and those that follow are in rationalized MKS units, the actual computations by IBM facilities were based on inches. This discrepancy disappears in the final results because only ratios are involved.

(4) Calculation of $\sum_{i=1}^{n} \sum_{j=1}^{r} \left(\frac{4\pi}{I} H_{ij} \right)^2$ for a selected list of coils. These calculations are the most tedious from the viewpoint of keeping details straight. For the sake of clarity the following subscript notation is adopted:

i - identification of a given turn (in a particular layer) whose field is being evaluated.

j - identification of the particular layer in which is located the given turn whose field is being evaluated.

i' and j' - similar identification for the remaining turns which contribute to the field of the given turn.

This notation is consistent with that given in the legend following Equation 14 and used in the headings of Table 13. The code i is numbered consecutively from one end of the coil to the other, and the codes j and j' from the innermost layer to the outermost. The code i' is numbered consecutively from the turn whose field is being explored ($i' = 1$ for this turn) to the last turn at the right-hand end. The following calculations for the largest coil are described in a manner consistent with the above notation. For any turn i in the layer j the square of the magnetic field may be written as

$$\left(\frac{4\pi}{I} H_{ij}\right)^2 = \left(\sum_{j'=1}^{4} \sum_{i'=i+1}^{71-i} \frac{4\pi}{I} H_{ri'j'j}\right)^2$$

$$+\left(\sum_{\substack{j'=1 \\ j' \neq j}}^{4} \frac{4\pi}{I} H_{z(i'=1)j'j} + 2 \sum_{j'=1}^{4} \sum_{i'=2}^{i} \frac{4\pi}{I} H_{zi'j'j}\right.$$

$$\left. + \sum_{j'=1}^{4} \sum_{i'=i+1}^{71-i} \frac{4\pi}{I} H_{zi'j'j}\right)^2 . \qquad (95)^*$$

This series of summations is so written as to take full advantage of the existing symmetries that were described earlier. Thus, because turns located axially symmetrically with respect to the turn in question contribute equal and opposite amounts to the H_r of that turn, all such cancellations are made and those terms omitted in the summation in the first pair of parentheses in Equation 95. Because turns located axially symmetrically with respect to the turn in question contribute equal amounts of like sign to the H_z of that turn, the second pair of parentheses includes three terms. The first term sums the influence of turns directly over and under the turn in question; the second term sums the influence of the symmetrical pairs of turns; and the third term sums the influence of the remaining turns.

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*The second term in the second pair of parentheses on the right-hand side is omitted when $i = 1$. When $i = 35$, $i + 1 = 71 - i$, and any single series between the limits $i + 1$ and $71 - i$ contains only one term, namely, that for $i' = 36$.

Since the above equation was written for the largest coil and since the remaining coils to be studied are identical to portions of the largest coil, the expressions for $\left(\frac{4\pi}{I} H_{ij}\right)^2$ for the remaining coils will contain corresponding portions of the terms in Equation 95. A general expression for the first 20 coils specified in the text, that is, those having a common inside diameter but differing in the number of layers is as follows:

$$\left(\frac{4\pi}{I} H_{ij}\right)^2 = \left(\sum_{j'=1}^{m} \sum_{i'=i+1}^{n+1-i} \frac{4\pi}{I} H_{ri'j'j}\right)^2 + \left(\sum_{\substack{j'=1 \\ j'\neq j}}^{m} \frac{4\pi}{I} H_{z(i'=1)j'j}\right.$$

$$\left. + 2 \sum_{j'=1}^{m} \sum_{i'=2}^{i} \frac{4\pi}{I} H_{zi'j'j} + \sum_{j'=1}^{m} \sum_{i'=i+1}^{n+1-i} \frac{4\pi}{I} H_{zi'j'j}\right)^2, \quad (96)$$

where m is the number of layers and n the number of turns per layer in any given coil. When $m = 4$ and $n = 70$, Equation 96 reverts to Equation 95. The second term in the second pair of parentheses on the right-hand side of Equation 96 is omitted when $i = 1$. When $i = \frac{n}{2}$, $i + 1 = n + 1 - i$, and the two series between the limits $i + 1$ and $n + 1 - i$ contain only one term each, namely, that for $i' = \frac{n}{2} + 1$. A general expression for the remaining 15 coils is as follows:

$$\left(\frac{4\pi}{I} H_{ij}\right)^2 = \left(\sum_{i'=i+1}^{n+1-i} \frac{4\pi}{I} H_{ri'j'j}\right)^2 + \left(2 \sum_{i'=2}^{i} \frac{4\pi}{I} H_{zi'j'j}\right.$$

$$+ \sum_{i' = i + 1}^{n + 1 - i} \frac{4\pi}{I} H_{zi'j'j} \Bigg)^2 , \qquad (97)$$

where $j' = j = 2$, 3, and 4, respectively, for the $2\frac{1}{2}$-, 3-, and $3\frac{1}{2}$-inch inside-diameter coils.

The quantity $\left(\frac{4\pi}{I} H_{ij}\right)^2$ is to be computed for every turn ($i = 1, 2, 3, ---, \frac{n}{2}$ and $j = 1$ to m) in half of each coil (as obtained when cutting a coil by a plane perpendicular to the axis and midway between the ends). Symmetry shows the other half of any coil to be an image of the first half. Then Equation 19 can be modified to read as follows:

$$\frac{R_{a-c}}{R_{d-c}} = 1 + F(x) + 2r^2\, G(X) \frac{\sum_{j=1}^{m}\left[b_j \sum_{i=1}^{n/2}\left(\frac{4\pi}{I} H_{ij}\right)^2\right]}{n \sum_{j=1}^{m} b_j} . \qquad (98)$$

When the summation

$$\sum_{i=1}^{n/2}\left(\frac{4\pi}{I} H_{ij}\right)^2$$

is performed for each of the 35 coils, Equation 98 is ready for final computation of the resistance ratio. The accompanying table gives the selected values of frequency and the corresponding values of the skin-effect number X and functions F(X) and G(X) that are to be used in Equation 98. Completion of these computations and tabulation of the computed data constitute the burden that was borne by IBM facilities at Iowa State College

Table 14. Functions for Study of Skin and Proximity Effects for #8 Copper

Frequency f cycles/sec.	Skin-Effect Number X	F(X)	G(X)
10	0.11042	0.0000007743	0.000002324
20	0.15618	0.000003099	0.000009296
50	0.2469	0.00001935	0.00005806
100	0.3492	0.00007744	0.0002322
200	0.4939	0.0003098	0.0009282
500	0.7809	0.001934	0.005750
1000	1.1042	0.007695	0.02229
2000	1.5618	0.03024	0.07952
5000	2.469	0.1685	0.2875
10,000	3.492	0.4891	0.4972
20,000	4.939	1.021	0.7444

For X 2.2, $F = \frac{5X^4}{960 + 4X^4}$

For X 1.7, $G = \frac{11X^4}{704 + 20X^4}$

$r = 0.1285 \times 2.54 \times 1/2 \times 0.01 = 0.00163195$ meter

$\mu = 4 \quad \times 10^{-7}$ weber per meter-ampere-turn

$\sigma = 5.800 \times 10^{7}$ mhos per meter

$X = r\sqrt{2\pi\mu f\sigma} = 0.03492\sqrt{f}$